Regional Mathematical Olympiad-2019 problems and solutions

1. Suppose x is a nonzero real number such that both x^5 and $20x + \dfrac{19}{x}$ are rational numbers. Prove that x is a rational number.

   Solution: Since x^5 is rational, we see that $(20x)^5$ and $(x/19)^5$ are rational numbers. But

   $$(20x)^5 - \left(\frac{19}{x}\right)^5 = \left(20x - \frac{19}{x}\right)\left((20x)^4 + (20^3 \cdot 19)x^2 + 20^2 \cdot 19^2 + (20 \cdot 19^3)\frac{1}{x^2} + \frac{19^4}{x^4}\right).$$

   Consider

   $$\begin{aligned} T &= \left((20x)^4 + (20^3 \cdot 19)x^2 + 20^2 \cdot 19^2 + (20 \cdot 19^3)\frac{1}{x^2} + \frac{19^4}{x^4}\right) \\ &= \left((20x)^4 + \frac{19^4}{x^4}\right) + 20 \cdot 19\left((20x)^2 + \frac{19^2}{x^2}\right) + (20^2 \cdot 19^2). \end{aligned}$$

   Using $20x + (19)/x$ is rational, we get

   $$(20x)^2 + \frac{19^2}{x^2} = \left(20x + \frac{19}{x}\right)^2 - 2 \cdot 20 \cdot 19$$

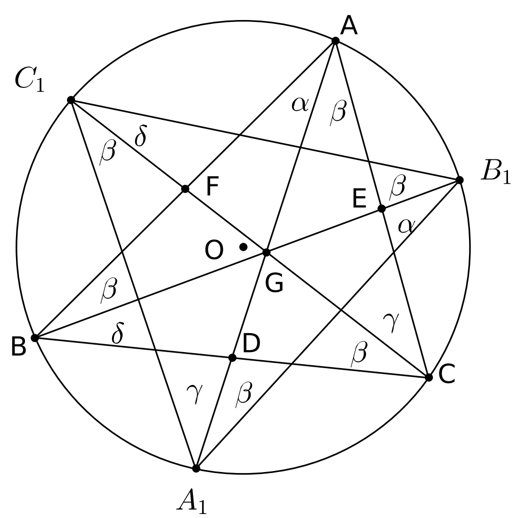
   is rational. This leads to

   $$(20x)^4 + \frac{19^4}{x^4} = \left((20x)^2 + \frac{19^2}{x^2}\right)^2 - 2 \cdot 20^2 \cdot 19^2$$

   is also rational. Thus T is a rational number and $T \neq 0$. We conclude that $20x - (19/x)$ is a rational number. This combined with the given condition that $20x + (19/x)$ is rational shows $2 \cdot 20 \cdot x$ is rational. Therefore x is rational.

2. Let ABC be a triangle with circuncircle Ω and let G be the centroid of triangle ABC. Extend AG, BG and CG to meet the circle Ω again in A_1, B_1 and C_1, respectively. Suppose $\angle BAC = \angle A_1B_1C_1$, $\angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that ABC and $A_1B_1C_1$ are equilateral triangles.

   Solution:



Let $\angle BAA_1 = \alpha$ and $\angle A_1AC = \beta$. Then $\angle BB_1A_1 = \alpha$. Using that angles at A and B_1 are same, we get $\angle BB_1C_1 = \beta$. Then $\angle C_1CB = \beta$. If $\angle ACC_1 = \gamma$, we see that $\angle C_1A_1A = \gamma$. Therefore $\angle AA_1B_1 = \beta$. Similarly, we see that $\angle B_1BA = \angle A_1C_1C = \beta$ and $\angle B_1BC = \angle B_1C_1C = \delta$.

Since $\angle FBG = \angle BCG = \beta$, it follows that FB is tangent to the circumcircle of $\triangle BGC$ at B. Therefore $FB^2 = FG \cdot FC$. Since $FA = FB$, we get $FA^2 = FG \cdot FC$. This implies that FA is tangent to the circumcircle of of $\triangle AGC$ at A. Therefore $\alpha = \angle GAF = \angle GCA = \gamma$. A similar analysis gives $\alpha = \delta$.

It follows that all the angles of $\triangle ABC$ are equal and all the angles of $\triangle A_1B_1C_1$ are equal. Hence ABC and $A_1B_1C_1$ are equilateral triangles.

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2+b^3+c^3} + \frac{b}{b^2+c^3+a^3} + \frac{c}{c^2+a^3+b^3} \leq \frac{1}{5abc}.$$

Solution: Observe that

$$a^2+b^3+c^3 = a^2(a+b+c) + b^3 + c^3 = (a^3+b^3+c^3) + a^2(b+c) \geq 3abc + a^2b + a^2c.$$

Hence

$$\frac{a}{a^2+b^3+c^3} \leq \frac{1}{3bc+ab+ac}.$$

Using AM-HM inequality, we also have

$$\frac{3}{bc} + \frac{1}{ca} + \frac{1}{ab} \geq \frac{25}{3bc+ca+ab}.$$

Thus we get

$$\frac{a}{a^2+b^3+c^3} \leq \frac{1}{3bc+ab+ac} \leq \frac{1}{25}\left(\frac{3}{bc} + \frac{1}{ca} + \frac{1}{ab}\right).$$

Similarly, we get

$$\frac{b}{b^2+c^3+a^3} \leq \frac{1}{25}\left(\frac{3}{ca} + \frac{1}{ab} + \frac{1}{bc}\right)$$

and

$$\frac{c}{c^2+a^3+b^3} \leq \frac{1}{25}\left(\frac{3}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)$$

Adding, we get

$$\begin{aligned}\frac{a}{a^2+b^3+c^3} + \frac{b}{b^2+c^3+a^3} + \frac{c}{c^2+a^3+b^3} &\leq \frac{5}{25}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \\ &= \frac{1}{5abc}.\end{aligned}$$

4. Consider the following 3×2 array formed by using the numbers $1, 2, 3, 4, 5, 6$:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

Observe that all row sums are equal, but the sum of the squares is not the same for each row. Extend the above array to a $3 \times k$ array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers $7, 8, 9, \ldots, 3k$ such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \quad \textbf{and} \quad \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2.$$

Solution: Consider the following extension:

$$\begin{pmatrix} 1 & 6 & 3+6 & 4+6 & 2+(2\cdot 6) & 5+(2\cdot 6) \\ 2 & 5 & 1+6 & 6+6 & 3+(2\cdot 6) & 4+(2\cdot 6) \\ 3 & 4 & 2+6 & 5+6 & 1+(2\cdot 6) & 6+(2\cdot 6) \end{pmatrix}$$

of

$$\begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

This reduces to

$$\begin{pmatrix} 1 & 6 & 9 & 10 & 14 & 17 \\ 2 & 5 & 7 & 12 & 15 & 16 \\ 3 & 4 & 8 & 11 & 13 & 18 \end{pmatrix}.$$

Observe

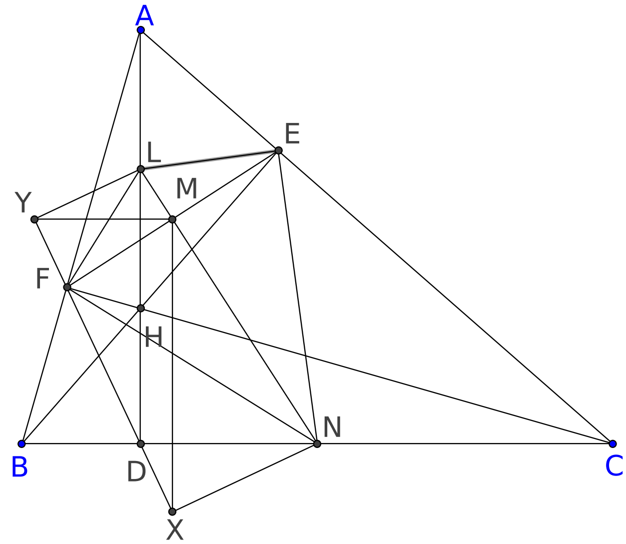
$$\begin{aligned} 1+6+9+10+14+17 &= 57; \quad 1^2+6^2+9^2+10^2+14^2+17^2 = 703; \\ 2+5+7+12+15+16 &= 57; \quad 2^2+5^2+7^2+12^2+15^2+16^2 = 703; \\ 3+4+8+11+13+18 &= 57; \quad 3^2+4^2+8^2+11^2+13^2+18^2 = 703. \end{aligned}$$

Thus, in the new array, all row sums are equal and the sum of the squares of entries in each row are the same. Here $k = 6$ and we have added numbers from 7 to 18.

5. In a triangle ABC, let H be the orthocenter, and let D, E, F be the feet of altitudes from A, B, C to the opposite sides, respectively. Let L, M, N be midpoints of segments AH, EF, BC, respectively. Let X, Y be feet of altitudes from L, N on to the line DF. Prove that XM is perpendicular to MY.

Solution: Observe that AFH and HEA are right-angled triangles and L is the mid-point of AH. Hence $LF = LA = LE$. Similarly, considering the right triangles BFC and BEC, we get $NF = NE$. Since M is the mid-point of FE it follows that $\angle LMF = \angle NMF = 90°$ and L, M, N are collinear. Since LY and NX are perpendiculars to XY, we conclude that $YFML$ and $FXNM$ are cyclic quadrilaterals. Thus

$$\angle FLM = \angle FYM, \quad \text{and} \quad \angle FXM = \angle FNM.$$



We also observe that CFB is a right triangle and N is the mid-point of BC. Hence $NF = NC$. We get

$$\angle NFC = \angle NCF = 90^\circ - \angle B.$$

Similarly, $LF = LA$ gives

$$\angle LFA = \angle LAF = 90^\circ - \angle B.$$

We obtain

$$\angle LFN = \angle LFC + \angle NFC = \angle LFC + 90 - \angle B = \angle LFC + \angle LFA = \angle AFC = 90^\circ.$$

In triangles YMX and LFN, we have

$$\angle XYM = \angle FYM = \angle FLM = \angle FLN,$$

and

$$\angle YXM = \angle FXM = \angle FNM = \angle FNL.$$

It follows that $\angle YMX = \angle LFN = 90^\circ$. Therefore $YM \perp MX$.

6. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that $\gcd(a, b) = \gcd(b, c) = \gcd(c, d) = \gcd(d, a) = 1$.

Solution: Let the given integers be $a_1, a_2, \ldots, a_{91}$. Take a 91×91 grid and color the cell at (i, j) black if $\gcd(a_i, a_j) = 1$. Then at least $2 \times 456 = 912$ cells are colored black. If d_i is the number of black cells in the ith column, then $\sum d_i \geq 912$. Now,

$$\begin{aligned}
\sum_{1}^{91} \binom{d_i}{2} &\geq \frac{1}{2}\left[\frac{1}{91}\left(\sum_{i=1}^{91} d_i\right)^2 - \sum_{i=1}^{91} d_i\right] \\
&= \frac{1}{2 \times 91}\left(\sum_{i=1}^{91} d_i\right)\left(\sum_{i=1}^{91} d_i - 91\right) \\
&\geq \frac{1}{2 \times 91} \times 2 \times 456 \times (2 \times 456 - 91) \\
&> \binom{91}{2}
\end{aligned}$$

Since there are only $\binom{91}{2}$ distinct pairs of columns, there must be at least one pair of rows (u, v) that occur with two distinct columns s, t. Thus $(u, s), (u, t), (v, s)$ and (v, t) are all black. Thus if the integers corresponding to the columns u, v, s, t are a, c, b, d respectively, then $\gcd(a, b) = \gcd(b, c) = \gcd(c, d) = \gcd(d, a) = 1$.

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