Solutions to Regional Mathematical Olympiad-2018
(Kerala Region and Tamil Nadu Region)

1. Let ABC be an acute-angled triangle and let D be an interior point of the line segment BC. Let the circumcircle of triangle ACD intersect AB at E (E between A and B) and let the circumcircle of triangle ABD intersect AC at F (F between A and C). Let O be the circumcentre of triangle AEF. Prove that OD bisects $\angle EDF$.

   Solution: Observe that $ACDE$ is a cyclic quadrilateral. Hence $\angle BDE = \angle EAC = \angle A$. Similarly, the cyclicity of $ABDF$ gives $\angle CDF = \angle FAB = \angle A$. We also have

   $$\angle EDF = 180^\circ - \angle BDE - \angle CDF = 180^\circ - 2\angle A.$$

   Since $\angle A$ is acute, O and D lie on the opposite sides of EF. Hence $\angle EOF = 2\angle A$. Therefore

   $$\angle EDF + \angle EOF = 180^\circ,$$

   so that $EOFD$ is a cyclic quadrilateral. Since $OE = OF$, we have $\angle OEF = \angle OFE$. Finally, the cyclicity of $EOFD$ gives

   $$\angle ODE = \angle OFE = \angle OEF = \angle ODF.$$

   This shows that OD bisects $\angle EDF$.

2. Find the set of all real values of a for which the real polynomial equation $P(x) = x^2 - 2ax + b = 0$ has real roots given that $P(0) \cdot P(1) \cdot P(2) \neq 0$ and $P(0)$, $P(1)$, $P(2)$ form a geometric progression.

   Solution: Since none of $P(0), P(1), P(2)$ is zero, we have $P(1)^2 = P(0)P(2)$. Observe that $P(0) = b$, $P(1) = 1-2a+b$ and $P(2) = 4-4a+b$. Therefore we obtain $(1-2a+b)^2 = b(4-4a+b)$. This simplifies to $(2a-1)^2 = 2b$. Now $P(x) = 0$ has real roots if and only if $4a^2 \geq 4b$. Hence $2a^2 \geq 2b = (2a-1)^2$. This reduces to $2a^2 - 4a + 1 \leq 0$. Consider the quadratic equation $2X^2 - 4X + 1 = 0$. This has two roots

   $$\alpha = \frac{2+\sqrt{2}}{2}, \quad \beta = \frac{2-\sqrt{2}}{2}.$$

   Hence $2a^2 - 4a + 1 \leq 0$ if and only if a lies in the interval $\left[\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right]$.

3. Show that there are infinitely many 4-tuples (a, b, c, d) of natural numbers such that $a^3 + b^4 + c^5 = d^7$.

   Solution: Observe that $\text{lcm}(3, 4, 5) = 60$. We look for a solution in the form $a^3 = b^4 = c^5 = l^{60k}$. We choose l, k such that the condition is satisfied. The given equation gives

   $$3l^{60k} = d^7.$$

   This suggests choosing $l = 3$ so that $3^{60k+1} = d^7$. Now we take care of k by choosing k such that 7 divides $60k + 1$. For example we can take $k = 5$ so that $301 = 7 \times 43$. Thus we get

   $$3^{300} + 3^{300} + 3^{300} = 3^{301}.$$

   Choose $a = 3^{100}$, $b = 3^{75}$, $c = 3^{60}$ and $d = 3^{43}$. Then we get

   $$a^3 + b^4 + c^5 = d^7.$$

This gives one solution. This suggests choosing $a = 3^{100} \cdot m^{140}$, $b = 3^{75} \cdot m^{105}$, $c = 3^{60} \cdot m^{84}$ and $d = 3^{43} \cdot m^{60}$. We see that

$$a^3 + b^4 + c^5 = 3^{300} \cdot m^{420} + 3^{300} \cdot m^{420} + 3^{300} \cdot m^{420} = m^{420} 3^{301} = (3^{43} \cdot m^{60})^7 = d^7.$$

We can give different values for m and get infinitely many solutions of the equation.

4. Suppose 100 points in the plane are coloured using two colours, red and white, such that each red point is the centre of a circle passing through at least three white points. What is the least possible number of white points?

Solution: Let n be the number of white points. Then we can draw at most $\binom{n}{3}$ circles. Thus the number of white and red points together is at most $n + \binom{n}{3}$. We observe that

$$9 + \binom{9}{3} = 93, \quad 10 + \binom{10}{3} = 130.$$

Thus $n \geq 10$. We show that $n = 10$ works.

Take any 10 points such that no three are collinear and no four are concyclic. Then we get $\binom{10}{3} = 120$ points as centre of distinct circles. Among these, there may be some points from the original 10 points. Even if we leave out these 10 points, we have at least 110 points which are centres of circles formed by the ten points we have chosen. Choose any 90 points from them and colour them red and colour the original 10 points white. We get 100 points of which 10 are white and remaining 90 are red. Each of these 90 red points is the centre of a circle passing through some three white points.

5. n a cyclic quadrilateral $ABCD$ with circumcentre O, the diagonals AC and BD intersect at X. Let the circumcircles of triangles AXD and BXC intersect at Y. Let the circumcircles of triangles AXB and CXD intersect at Z. If O lies inside $ABCD$ and if the points O, X, Y, Z are all distinct, prove that O, X, Y, Z lie on a circle.

Solution: Observe that

$$\begin{aligned} \angle AYB &= 360^\circ - \angle AYX - \angle XYB \\ &= (180^\circ - \angle AYX) + (180^\circ - XYB) \\ &= \angle XDA + \angle XCB = 2\angle ACB = \angle AOB. \end{aligned}$$

It follows that A, B, O, Y are concyclic. Similarly,

$$\begin{aligned} \angle CZB &= 360^\circ - \angle CZX - \angle XZB \\ &= (180^\circ - \angle CZX) + (180 - \angle XZB) \\ &= \angle CDX + \angle XAB = \angle COB, \end{aligned}$$

so that C, Z, O, B are concyclic. We have

$$\begin{aligned} \angle XYO &= 360^\circ - \angle XYA - \angle AYO \\ &= (180^\circ - \angle XYA) + (180^\circ - \angle AYO) \\ &= \frac{\angle AOB}{2} + \left(90^\circ - \frac{\angle AOB}{2}\right) = 90^\circ. \end{aligned}$$

Similarly, we can show that $\angle XZO = 90^\circ$. It follows that X, Y, O, Z are concyclic.

6. Define a sequence $\langle a_n \rangle_{n\geq 1}$ of real numbers by

$$a_1 = 2, \quad a_{n+1} = \frac{a_n^2+1}{2}, \text{ for } n \geq 1.$$

Prove that

$$\sum_{j=1}^{N} \frac{1}{a_j+1} < 1$$

for every natural number N.

Solution: If $N = 1$, then the sum is just $1/(a_1+1) = 1/3 < 1$. Hence we may assume that $N > 1$. It is easy to see recursively (that is using induction) that $a_j > 1$ for all j. We observe that

$$a_{j+1} - a_j = \frac{a_j^2+1}{2} - a_j = \frac{(a_j-1)^2}{2};$$

$$a_{j+1} - 1 = \frac{a_j^2+1}{2} - 1 = \frac{a_j^2-1}{2} = \frac{(a_j-1)(a_j+1)}{2}.$$

Hence

$$\frac{1}{a_j+1} = \frac{1}{2}\left(\frac{a_j-1}{a_{j+1}-1}\right),$$

for all $j \geq 1$. Therefore

$$\begin{aligned}
\sum_{j=1}^{N} \frac{1}{a_j+1} = \frac{1}{2}\sum_{j=1}^{N} \frac{a_j-1}{a_{j+1}-1} &= \frac{1}{2}\sum_{j=1}^{N} \frac{(a_j-1)^2}{(a_{j+1}-1)(a_j-1)} \\
&= \sum_{j=1}^{N} \frac{a_{j+1}-a_j}{(a_{j+1}-1)(a_j-1)} \\
&= \sum_{j=1}^{N} \left(\frac{1}{a_j-1} - \frac{1}{a_{j+1}-1}\right) \\
&= \frac{1}{a_1-1} - \frac{1}{a_{N+1}-1} < \frac{1}{a_1-1} = 1.
\end{aligned}$$

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