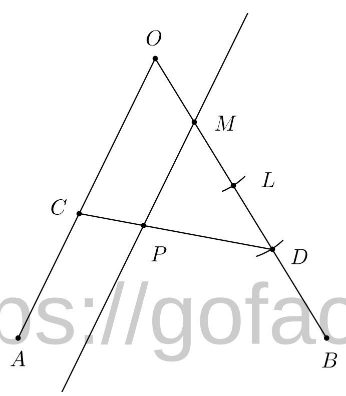
Regional Mathematical Olympiad-2017

Solutions

1. Let AOB be a given angle less than $180°$ and let P be an interior point of the angular region determined by $\angle AOB$. Show, with proof, how to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray OA and D lies on the ray OB, and $CP : PD = 1 : 2$.

   Solution: Draw a line parallel to OA through P. Let it intersect OB in M. Using compasses, draw an arc of a circle with centre M and radius MO to cut OB in L, $L \neq O$. Again with L as centre and with the same radius OM draw one more arc of a circle to cut OB in D, $D \neq M$. Join DP and extend it to meet OA in C. Then CD is the required line segment such that $CP : PD = 1 : 2$. This follows from similar triangles OCD and MPD.



2. Show that the equation

   $$a^3+(a+1)^3+(a+2)^3+(a+3)^3+(a+4)^3+(a+5)^3+(a+6)^3 = b^4+(b+1)^4$$

   has no solutions in integers a, b.

   Solution: We use divisibility argument by 7. Observe that the remainders of seven consecutive cubes modulo 7 are $0, 1, 1, 6, 1, 6, 6$ in some (cyclic) order. Hence the sum of seven consecutive cubes is 0 modulo 7. On the other hand the remainders of two consecutive fourth powers modulo 7 is one of the sets $\{0, 1\}$, $\{1, 2\}$, $\{2, 4\}$, $\{4, 4\}$. Hence the sum of two fourth powers is never divisible by 7. It follows that the given equation has no solution in integers.

3. Let $P(x) = x^2 + \frac{1}{2}x + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q\big(P(x)\big)$ for all real x. Find all the real roots of $P\big(Q(x)\big) = 0$.

Solution: Observe that

$$P(x)Q(x) = x^4 + \left(c + \frac{1}{2}\right)x^3 + \left(b + \frac{c}{2} + d\right)x^2 + \left(\frac{d}{2} + bc\right)x + bd.$$

Similarly,

$$\begin{aligned} Q\big(P(x)\big) &= \left(x^2 + \frac{1}{2}x + b\right)^2 + c\left(x^2 + \frac{1}{2}x + b\right) + d \\ &= x^4 + x^3 + \left(2b + \frac{1}{4} + c\right)x^2 + \left(b + \frac{c}{2}\right)x + b^2 + bc + d. \end{aligned}$$

Equating coefficients of corresponding powers of x, we obtain

$$c + \frac{1}{2} = 1, \quad b + \frac{c}{2} + d = 2b + \frac{1}{4} + c, \quad \frac{d}{2} + bc = b + \frac{c}{2}, \quad b^2 + bc + d = bd.$$

Solving these, we obtain

$$c = \frac{1}{2}, d = 0, b = \frac{-1}{2}.$$

Thus the polynomials are

$$P(x) = x^2 + \frac{1}{2}x - \frac{1}{2}, \quad Q(x) = x^2 + \frac{1}{2}x.$$

Therefore,

$$\begin{aligned} P\big(Q(x)\big) &= \left(x^2 + \frac{1}{2}x\right)^2 + \frac{1}{2}\left(x^2 + \frac{1}{2}x\right) - \frac{1}{2} \\ &= x^4 + x^3 + \frac{3}{4}x^2 + \frac{1}{4}x - \frac{1}{2}. \end{aligned}$$

It is easy to see that

$$P(Q(-1)) = 0, \quad P(Q(1/2)) = 0.$$

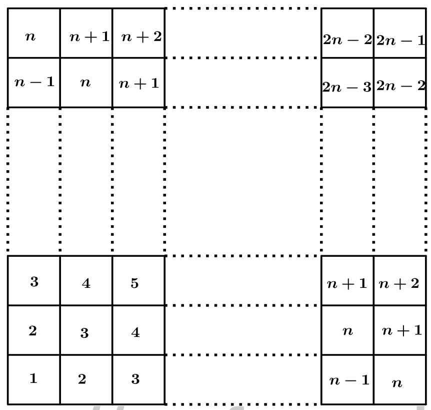
Thus $(x+1)$ and $(x - 1/2)$ are factors of $P(Q(x))$. The remaining factor is

$$h(x) = x^2 + \frac{1}{2}x + 1.$$

The discriminant of $h(x)$ is $D = (1/4) - 4 < 0$. Hence $h(x) = 0$ has no real roots. Therefore the only real roots of $P(Q(x)) = 0$ are -1 and $1/2$.

4. Consider n^2 unit squares in the xy-plane centred at point (i, j) with integer coordinates, $1 \leq i \leq n$, $1 \leq j \leq n$. It is required to colour each unit square in such a way that whenever $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, the three squares with centres at $(i, k), (j, k), (j, l)$ have distinct colours. What is the least possible number of colours needed?

Solution: We first show that at least $2n - 1$ colours are needed. Observe that squares with centres $(i, 1)$ must all have diffrent colours for $1 \leq i \leq n$; let us call them $C_1, C_2, \ldots, C_n$. Besides, the squares with centres (n, j), for $2 \leq j \leq n$ must have different colours and these must be different from $C_1, C_2, \ldots, C_n$. Thus we need at least $n + (n - 1) = 2n - 1$ colours. The following diagram shows that $2n - 1$ colours will suffice.



5. Let Ω be a circle with a chord AB which is not a diameter. Let Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D. Likewise, let Γ_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω.

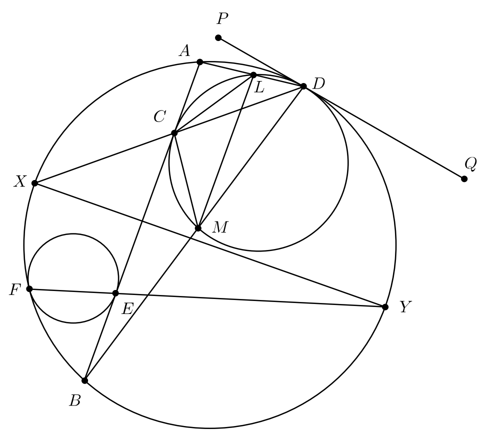
Solution: Draw the tangent PQ at D such that D is between P and Q. Join D to A, B and C. Let $L = DA \cap \Gamma_1$ and $M = DB \cap \Gamma_1$. Join C to L and M. Observe that

$$\angle ADP = \angle LMD = \angle ABD. \tag{1}$$

Therefore LM is parallel to AB and hence $\angle LMC = \angle MCB$ (alternate angles). Again observe that

$$\angle ADC = \angle LDC = \angle LMC = \angle MCB = \angle MDC = \angle BDC. \tag{2}$$

Thus CD bisects $\angle ADB$. Hence X is the midpoint of the arc AB not containing D. Similarly Y is the midpoint of the arc AB not containing F. Thus the arc XY is half of the sum of two arcs that together constitute the circumference of Ω and hence it is a diameter.



6. Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1}+\frac{y+1}{z+1}+\frac{z+1}{x+1} \leq \frac{x-1}{y-1}+\frac{y-1}{z-1}+\frac{z-1}{x-1}.$$

Solution: We may assume that $x = \max\{x, y, z\}$. There are two cases: $x \geq y \geq z$ and $x \geq z \geq y$. We consider both these cases. The inequality is equivalent to

$$\left\{\frac{x-1}{y-1}-\frac{x+1}{y+1}\right\}+\left\{\frac{y-1}{z-1}-\frac{y+1}{z+1}\right\}+\left\{\frac{z-1}{x-1}-\frac{z+1}{x+1}\right\} \geq 0.$$

This is further equivalent to

$$\frac{x-y}{y^2-1}+\frac{y-z}{z^2-1}+\frac{z-x}{x^2-1} \geq 0.$$

Suppose $x \geq y \geq z$. We write

$$\frac{x-y}{y^2-1}+\frac{y-z}{z^2-1}+\frac{z-x}{x^2-1}=\frac{x-y}{y^2-1}+\frac{y-z}{z^2-1}+\frac{z-y+y-x}{x^2-1}.$$

This reduces to

$$(x-y)\frac{(x^2-y^2)}{(x^2-1)(y^2-1)}+(y-z)\frac{(x^2-z^2)}{(x^2-1)(z^2-1)}.$$

Since $x \geq y$ and $x \geq z$, this sum is nonnegative.

Suppose $x \geq z \geq y$. We write

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} = \frac{x-z+z-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1}.$$

This reduces to

$$(x-z)\frac{(x^2-y^2)}{(x^2-1)(y^2-1)} + (z-y)\frac{(z^2-y^2)}{(y^2-1)(z^2-1)}.$$

Since $x \geq z$ and $z \geq y$, this sum is nonnegative.

Thus

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} \geq 0$$

in both the cases. This completes the proof.

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