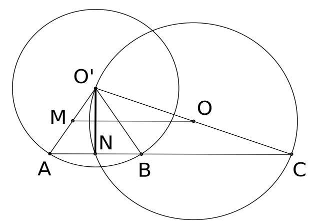
CRMO-2015 questions and solutions

1. Two circles Γ and Σ, with centres O and O', respectively, are such that O' lies on Γ. Let A be a point on Σ and M the midpoint of the segment AO'. If B is a point on Σ different from A such that AB is parallel to OM, show that the midpoint of AB lies on Γ.

   Solution: Let C be the reflection of O' with respect to O. Then in triangle $O'AC$, the midpoints of the segments $O'A$ and $O'C$ are M and O, respectively. This implies AC is parallel to OM, and hence B lies on AC. Let the line AC intersect Γ again at N. Since $O'C$ is a diameter of Γ it follows that $\angle O'NC = 90°$. Since $O'A = O'B$, we can now conclude that N is the midpoint of the segment AB.



2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a and b are real numbers. Suppose $\langle P(-1)^2, P(0)^2, P(1)^2 \rangle$ is an arithmetic progression of integers. Prove that a and b are integers.

   Solution: Observe that

$$P(-1) = 1 - a + b, \quad P(0) = b, \quad P(1) = 1 + a + b.$$

   The given condition gives

$$2b^2 = (1 - a + b)^2 + (1 + a + b)^2 = 2(1 + b)^2 + 2a^2 = 2 + 4b + 2b^2 + 2a^2.$$

   Hence $a^2 + 2b + 1 = 0$. Observe

$$1 + a^2 + b^2 + 2a + 2b + 2ab = (1 + a + b)^2 \in \mathbb{Z}.$$

   But $1, b^2, 2a^2 + 4b$ are all integers. Hence $4a + 4ab \in \mathbb{Z}$. This gives $16a^2(1 + b)^2$ is an integer. But $a^2 = -(2b + 1)$. Hence $16(2b + 1)(1 + b)^2$ is an integer. But

$$16(2b + 1)(1 + b)^2 = 16(1 + 4b + 5b^2 + 2b^3).$$

   Hence $16b(4 + 2b^2)$ is an integer. If $b = 0$, then b is an integer. Otherwise, this shows that b is a rational number. Because $b^2 \in \mathbb{Z}$, it follows that b is an integer. Since $a^2 = -(2b + 1)$, we get that a^2 is an integer. Now $4a(1 + b) \in \mathbb{Z}$. If $b \neq -1$, then a is rational and hence a is an integer. If $b = -1$, then we see that $P(-1) = -a$, $P(0) = b = -1$ and $P(1) = a$. Hence $a^2, 1, a^2$ is an AP. This implies that $a^2 = 1$ and hence $a = \pm 1$.

3. Show that there are infinitely many triples (x, y, z) of integers such that $x^3 + y^4 = z^{31}$.

   Solution: Choose $x = 2^{4r}$ and $y = 2^{3r}$. Then the left side is 2^{12r+1}. If we take $z = 2^k$, then we get $2^{12r+1} = 2^{31k}$. Thus it is sufficient to prove that the equation $12r + 1 = 31k$ has infinitely many solutions in integers. Observe that $(12 \times 18) + 1 = 31 \times 7$. If we choose $r = 31l + 18$ and $k = 12l + 7$, we get

$$12(31l + 18) + 1 = 31(12l + 7),$$

   for all l. Choosing $l \in \mathbb{N}$, we get infinitely many $r = 31l + 18$ and $k = 12l + 7$ such that $12r + 1 = 31k$. Going back we have infinitely many (x, y, z) of integers satisfying the given equation.

4. Suppose 36 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

   Solution: One can choose 3 objects out of 36 objects in $\binom{36}{3}$ ways. Among these choices all would be together in 36 cases; exactly two will be together in 36×32 cases. Thus three objects can be chosen such that no two adjacent in $\binom{36}{3} - 36 - (36 \times 32)$ ways. Among these, furthrer, two objects will be diametrically opposite in 18 ways and the third would be on either semicircle in a non adjacent portion in $36 - 6 = 30$ ways. Thus required number is

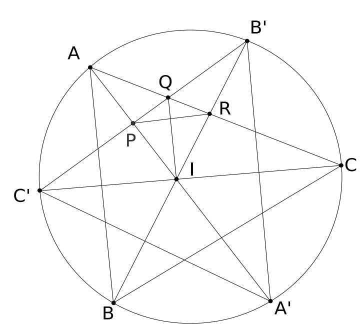
$$\binom{36}{3} - 36 - (36 \times 32) - (18 \times 30) = 5412.$$

5. Let ABC be a triangle with circumcircle Γ and incentre I. Let the internal angle bisectors of $\angle A$, $\angle B$ and $\angle C$ meet Γ in A', B' and C' respectively. Let $B'C'$ intersect AA' in P and AC in Q, and let BB' intersect AC in R. Suppose the quadrilateral $PIRQ$ is a kite; that is, $IP = IR$ and $QP = QR$. Prove that ABC is an equilateral triangle.

   Solution: We first show that AA' is perpendicular to $B'C'$. Observe $\angle C'A'A = \angle C'CA = \angle C/2$; $\angle A'C'C = \angle A'AC = \angle A/2$; and $\angle CC'B' = \angle CBB' = \angle B/2$. Hence

$$\angle C'AP + \angle AC'P = \angle C'AB + \angle BAP + \angle AC'P = \frac{\angle C}{2} + \frac{\angle A}{2} + \frac{\angle B}{2} = 90^\circ.$$

   It follows that $\angle APC' = \angle A'PC' = 90^\circ$. Thus $\angle IPQ = 90^\circ$. Since $PIRQ$ is a kite, we observe that $\angle IPR = \angle IRP$ and $\angle QPR = \angle QRP$. This implies that $\angle IRQ = 90^\circ$. Hence the kite $IRQP$ is also a cyclic quadrilateral. Since $\angle IRQ = 90^\circ$, we see that $BB' \perp AC$. Since BB' is the bisector of $\angle B$, we conclude that $\angle A = \angle C$.



   We also observe that the triangles IRC and IPB' are congruent triangles: they are similar, since $\angle IRC = \angle IPB' = 90^\circ$ and $\angle ICR = \angle C/2 = \angle IB'P (= \angle BCC')$; besides $IR = IP$. Therefore $IC = IB'$. But $B'I = B'C$. Thus $IB'C$ is an equilateral triangle. This means $\angle B'IC = 60^\circ$ and hence $\angle ICR = 30^\circ$. Therefore $\angle C/2 = 30^\circ$. Hence $\angle A = \angle C = 60^\circ$. It follows that ABC is equilateral.

6. Show that there are infinitely many positive real numbers a which are not integers such that $a(a - 3\{a\})$ is an integer. (Here $\{a\}$ denotes the fractional part of a. For example $\{1.5\} = 0.5$; $\{-3.4\} = 0.6$.)

Solution: We show that for each integer $n \geq 0$, the interval $(n, n+1)$ contains a such that $a(a - 3\{a\})$ is an integer. Put $a = n + f$, where $0 < f < 1$. Then $(n+f)(n-2f)$ must be an integer. This means $2f^2 + nf$ must be an integer. Since $0 < f < 1$, we must have $0 < 2f^2 + nf < 2 + n$. Hence $2f^2 + nf \in \{1, 2, 3, \ldots, n+1\}$. Taking $2f^2 + nf = 1$, we get a quadratic equation:

$$2f^2 + nf - 1 = 0.$$

Hence

$$f = \frac{-n + \sqrt{n^2+8}}{4}, \text{ and } a = n + \frac{-n + \sqrt{n^2+8}}{4}.$$

Thus we see that each a in the set

$$\left\{ n + \frac{-n + \sqrt{n^2+8}}{4} \; : \; n \in \mathbb{N} \right\}$$

is a real number, which is not an integer, such that $a(a - 3\{a\})$ is an integer.

Remark: Each interval $(n, n+1)$ contains $n+1$ such numbers, for $n \geq 0$, n an integer.

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