Solutions to RMO-2014 problems

1. In an acute-angled triangle ABC, $\angle ABC$ is the largest angle. The perpendicular bisectors of BC and BA intersect AC at X and Y respectively. Prove that circumcentre of triangle ABC is incentre of triangle BXY.

   Solution: Let D and E be the mid-points of BC and AB respectively. Since X lies on the perpendicular bisector of BC, we have $XB = XC$. Since $XD \perp BC$ and $XB = XC$, it follows that XD bisects $\angle BXC$. Similarly, YE bisects $\angle BYA$. Hence the point of intersection of XD and YE is the incentre of $\triangle BXY$. But this point of intersection is also the circumcentre of $\triangle ABC$, being the point of intersection of perpendicular bisectors of BC and AB.

2. Let x, y, z be positive real numbers. Prove that

$$\frac{y^2+z^2}{x} + \frac{z^2+x^2}{y} + \frac{x^2+y^2}{z} \geq 2(x+y+z).$$

   Solution: We write the inequality in the form

$$\frac{x^2}{y} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{y} + \frac{z^2}{x} + \frac{x^2}{z} \geq 2(x+y+z).$$

   We observe that $x^2 + y^2 \geq 2xy$. Hence $x^2 + y^2 - xy \geq xy$. Multiplying both sides by $(x+y)$, we get

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) \geq (x+y)xy.$$

   Thus

$$\frac{x^2}{y} + \frac{y^2}{x} \geq x + y.$$

   Similarly, we obtain

$$\frac{y^2}{z} + \frac{z^2}{y} \geq y + z, \quad \frac{z^2}{x} + \frac{x^2}{z} \geq x + y.$$

   Adding three inequalities, we get the required result.

3. Find all pairs of (x, y) of positive integers such that $2x + 7y$ divides $7x + 2y$.

   Solution: Let $d = \gcd(x, y)$. Then $x = dx_1$ and $y = dy_1$. We observe that $2x + 7y$ divides $7x + 2y$ if and only if $2x_1 + 7y_1$ divides $7x_1 + 2y_1$. This means $2x_1 + 7y_1$ should divide $49x_1 + 14y_1$. But $2x_1 + 7y_1$ divides $4x_1 + 14y_1$. Hence $2x_1 + 7y_1$ divides $45x_1$. Similarly, we can show that $2x_1 + 7y_1$ divides $45y_1$. Hence $2x_1 + 7y_1$ divides $\gcd(45x_1, 45y_1) = 45\gcd(x_1, y_1) = 45$. Hence

$$2x_1 + 7y_1 = 9, 15 \text{ or } 45.$$

If $2x_1 + 7y_1 = 9$, then $x_1 = 1$, $y_1 = 1$. Similarly, $2x_1 + 7y_1 = 15$ gives $x_1 = 4$, $y_1 = 1$. If $2x_1 + 7y_1 = 45$, then we get

$$(x_1, y_1) = (19, 1), (12, 3), (5, 5).$$

Thus all solutions are of the form

$$(x, y) = (d, d), (4d, d), (19d, d), (12d, 3d), (5d, 5d).$$

4. For any positive integer $n > 1$, let $P(n)$ denote the largest prime not exceeding n. Let $N(n)$ denote the next prime larger than $P(n)$. (For example $P(10) = 7$ and $N(10) = 11$, while $P(11) = 11$ and $N(11) = 13$.) If $n+1$ is a prime number, prove that the value of the sum

$$\frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)} = \frac{n-1}{2n+2}.$$

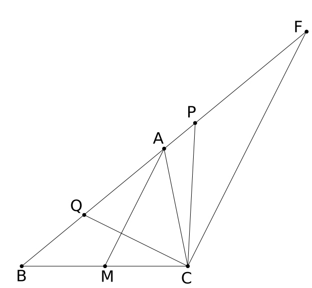
Solution: Let p and q be two consecutive primes, $p < q$. If we take any n such that $p \le n < q$, we see that $P(n) = p$ and $N(n) = q$. Hence the term $\frac{1}{pq}$ occurs in the sum $q - p$ times. The contribution from such terms is $\frac{q-p}{pq} = \frac{1}{p} - \frac{1}{q}$. Since $n+1$ is prime, we obtain

$$\frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)}$$
$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{p} - \frac{1}{n+1}\right) = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2n+2}.$$

Here p is used for the prime preceeding $n+1$.

5. Let ABC be a triangale with $AB > AC$. Let P be a point on the line AB beyond A such that $AP + PC = AB$. Let M be the mid-point of BC and let Q be the point on the side AB such that $CQ \perp AM$. Prove that $BQ = 2AP$.

Solution: Extend BP to F such $PF = PC$. Then $AF = AP + PF = AP + PC = AB$. Hence A is the mid-point of BF. Since M is the mid-point of BC, it follows that $AM \parallel FC$. But $AM \perp CQ$. Hence $FC \perp CQ$ at C. Therefore QCF is a rigt-angled triangle. Since $PC = PF$, it follows that $\angle PCF = \angle PFC$. Hence $\angle PQC = \angle PCQ$ which gives $PQ = PC = PF$. This implies that P is the mid-point of QF.



Thus we have $AP + AQ = PF$ and $BQ + QA = AP + PF$. This gives

$$2AP + AQ = PF + AP = BQ + QA.$$

We conclude that $BQ = 2AP$.

6. Suppose n is odd and each square of an $n \times n$ grid is arbitrarily filled with either by 1 or by -1. Let r_j and c_k denote the product of all numbers in j-th row and k-th column respectively, $1 \leq j, k \leq n$. Prove that

$$\sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k \neq 0.$$

Solution: Suppose we change $+1$ to -1 in a square. Then the product of the numbers in that row changes sign. Similarly, the product of numbers in the column also changes sign. Hence the sum

$$S = \sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k$$

decreases by 4 or increases by 4 or remains same. Hence the new sum is congruent to the old sum modulo 4. Let us consider the situation when all the squares have $+1$. Then $S = n + n = 2n = 2(2m+1) = 4m+2$. This means the sum S is is always of the form $4l + 2$ for any configuration. Therefore the sum is not equal to 0.

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