INMO 2005: Problems and Solutions

1. Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If $AK{=}KL{=}LM$, prove that the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.

Solution:
Let I be the incentre of triangle ABC and D be its projection on BC. Observe that $AB \neq AC$ as $AB = AC$ implies that $D = L = M$. So assume that $AC > AB$. Let N be the projection of I on KL. Then the perpendicular IN from I to KL is a bisector of KL and as $AK = LM$, it is a bisector of AM also. Hence $AI = IM$.

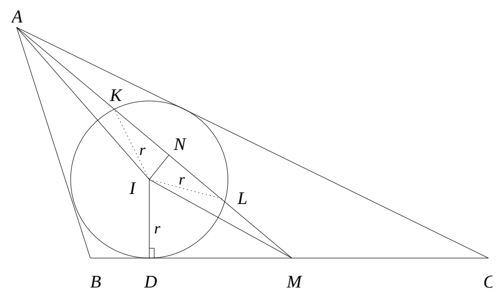

Fig. 1.

But $AI = \dfrac{r}{\sin\left(A/2\right)} = r \text{ cosec } \left(A/2\right)$ and

$$
\begin{aligned}
IM^2 &= ID^2 + DM^2 = r^2 + (BM - BD)^2 \\
&= r^2 + \left(\frac{a}{2} - (s-b)\right)^2.
\end{aligned}
$$

Hence $r^2 \text{ cosec}^2\left(A/2\right) = r^2 + \left((a/2) - (s-b)^2\right)^2$ giving $r^2 \cot^2\left(A/2\right) = \left((b-c)/2\right)^2$. Since $b > c$, we obtain $r \cot \left(A/2\right) = \left((b-c)/2\right)$. So $s - a = \left((b-c)/2\right)$. This gives $a = 2c$.
As $KN = NL$ and $AK = KL = LM$, we have $NL = AM/6$. We also have $AN = NM$. Now

$$
\begin{aligned}
r^2 = IL^2 = IN^2 + NL^2 &= AI^2 - AN^2 + NL^2 \\
&= AI^2 - \frac{1}{4}m_a^2 + \frac{1}{36}m_a^2 \\
&= r^2 \text{ cosec}^2\left(A/2\right) - \frac{2}{9}m_a^2.
\end{aligned}
$$

Hence $r^2 \cot^2\left(A/2\right) = \frac{2}{9}m_a^2$. From the above,we get

$$
\left(\frac{b-c}{2}\right)^2 = \frac{2}{9}\cdot\frac{1}{4}\left(2b^2 + 2c^2 - a^2\right).
$$

Simplification gives $5b^2 + 13c^2 - 18bc = 0$. This can be written as $(b-c)(5b-13c) = 0$. As $b \neq c$, we get $5b - 13c = 0$. To conclude, $a = 2c, 5b = 13c$ yield

$$
\frac{a}{10} = \frac{b}{13} = \frac{c}{5}.
$$

2. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of β.

Solution:
We have

$$\frac{77}{17} < \frac{\beta}{\alpha} < \frac{197}{43}.$$

That is,

$$4 + \frac{9}{17} < \frac{\beta}{\alpha} < 4 + \frac{25}{43}.$$

Thus $4 < \frac{\beta}{\alpha} < 5$. Since α and β are positive integers, we may write $\beta = 4\alpha + x$, where $0 < x < \alpha$. Now we get

$$4 + \frac{9}{17} < 4 + \frac{x}{\alpha} < 4 + \frac{25}{43}.$$

So $\frac{9}{17} < \frac{x}{\alpha} < \frac{25}{43}$; that is, $\frac{43x}{25} < \alpha < \frac{17x}{9}$.
We find the smallest value of x for which α becomes a well-defined integer. For $x = 1, 2, 3$ the bounds of α are respectively $\left(1\frac{18}{25}, 1\frac{8}{9}\right)$, $\left(3\frac{11}{25}, 3\frac{7}{9}\right)$, $\left(5\frac{4}{9}, 5\frac{2}{3}\right)$. None of these pairs contain an integer between them.
For $x = 4$, we have $\frac{43x}{25} = 6\frac{12}{25}$ and $\frac{17x}{9} = 7\frac{5}{9}$. Hence, in this case $\alpha = 7$, and $\beta = 4\alpha + x = 28 + 4 = 32$. This is also the least possible value, because, if $x \geq 5$, then $\alpha > \frac{43x}{25} \geq \frac{43}{5} > 8$, and so $\beta > 37$. Hence the minimum possible value of β is 32.

3. Let p, q, r be positive real numbers, not all equal, such that some two of the equations

$$px^2 + 2qx + r = 0, \quad qx^2 + 2rx + p = 0, \quad rx^2 + 2px + q = 0,$$

have a common root, say α. Prove that

(a) α is real and negative; and
(b) the third equation has non-real roots.

Solution:
Consider the discriminants of the three equations

$$\begin{aligned} px^2 + qr + r &= 0 && (1)\\ qx^2 + rx + p &= 0 && (2)\\ rx^2 + px + q &= 0. && (3) \end{aligned}$$

Let us denote them by D_1, D_2, D_3 respectively. Then we have

$$D_1 = 4(q^2 - rp), D_2 = 4(r^2 - pq), D_3 = 4(p^2 - qr).$$

We observe that

$$\begin{aligned} D_1 + D_2 + D_3 &= 4(p^2 + q^2 + r^2 - pq - qr - rp) \\ &= 2\left\{(p-q)^2 + (q-r)^2 + (r-p)^2\right\} > 0 \end{aligned}$$

since p, q, r are not all equal. Hence at least one of D_1, D_2, D_3 must be positive. We may assume $D_1 > 0$.

Suppose $D_2 < 0$ and $D_3 < 0$. In this case both the equations (2) and (3) have only non-real roots and equation (1) has only real roots. Hence the common root α must be between (2) and (3). But then $\bar{\alpha}$ is the other root of both (2) and (3). Hence it follows that (2) and (3) have same set of roots. This implies that

$$\frac{q}{r} = \frac{r}{p} = \frac{p}{q}.$$

Thus $p = q = r$ contradicting the given condition. Hence both D_2 and D_3 cannot be negative. We may assume $D_2 \geq 0$. Thus we have

$$q^2 - rp > 0,\ r^2 - pq \geq 0.$$

These two give

$$q^2 r^2 > p^2 qr$$

since p, q, r are all positive. Hence we obtain $qr > p^2$ or $D_3 < 0$. We conclude that the common root must be between equations (1) and (2).
Thus

$$\begin{aligned} p\alpha^2 + q\alpha + r &= 0 \\ q\alpha^2 + r\alpha + p &= 0 \end{aligned}$$

Eliminating α^2, we obtain

$$2(q^2 - pr)\alpha = p^2 - qr.$$

Since $q^2 - pr > 0$ and $p^2 - qr < 0$, we conclude that $\alpha < 0$.

The condition $p^2 - qr < 0$ implies that the equation (3) has only non-real roots.

Alternately one can argue as follows. Suppose α is a common root of two equations, say, (1) and (2). If α is non-real, then $\bar{\alpha}$ is also a root of both (1) and (2). Hence The coefficients of (1) and (2) are proportional. This forces $p = q = r$, a contradiction. Hence the common root between any two equations cannot be non-real. Looking at the coefficients, we conclude that the common root α must be negative. If (1) and (2) have common root α, then $q^2 \geq rp$ and $r^2 \geq pq$. Here at least one inequality is strict for $q^2 = pr$ and $r^2 = pq$ forces $p = q = r$. Hence $q^2 r^2 > p^2 qr$. This gives $p^2 < qr$ and hence (3) has nonreal roots.

4. All possible 6-digit numbers, in each of which the digits occur in **non-increasing** order (from left to right, e.g., 877550) are written as a sequence in **increasing** order. Find the 2005-th number in this sequence.

Solution I:
Consider a 6-digit number whose digits from left to right are in non increasing order. If 1 is the first digit of such a number, then the subsequent digits cannot exceed 1. The set of all such numbers with initial digit equal to 1 is

$$\{100000, 110000, 111000, 111100, 111110, 1111111\}.$$

There are elements in this set.

Let us consider 6-digit numbers with initial digit 2. Starting form 200000, we can go up to 222222. We count these numbers as follows:

200000	-	211111	:	6
220000	-	221111	:	5
222000	-	222111	:	4
222200	-	222211	:	3
222220	-	222221	:	2
222222	-	222222	:	1

The number of such numbers is 21. Similarly we count numbers with initial digit 3; the sequence starts from 300000 and ends with 333333. We have

300000	-	322222	:	21
330000	-	332222	:	15
333000	-	333222	:	10
333300	-	333322	:	6
333330	-	333332	:	3
333333	-	333333	:	1

We obtain the total number of numbers starting from 3 equal to 56. Similarly,

400000	-	433333	:	56
440000	-	443333	:	35
444000	-	444333	:	20
444400	-	444433	:	10
444440	-	444443	:	4
444444	-	444444	:	1
				126

500000	-	544444	:	126
550000	-	554444	:	70
555000	-	555444	:	35
555500	-	555544	:	15
555550	-	555554	:	5
555555	-	555555	:	1
				252

600000	-	655555	:	252
660000	-	665555	:	126
666000	-	666555	:	56
666600	-	666655	:	21
666660	-	666665	:	6
666666	-	666666	:	1
				462

700000	-	766666	:	462
770000	-	776666	:	210
777000	-	777666	:	84
777700	-	777766	:	28
777770	-	777776	:	7
777777	-	777777	:	1
				792

Thus the number of 6-digit numbers where digits are non-increasing starting from 100000 and ending with 777777 is

$$792 + 462 + 252 + 126 + 56 + 21 + 6 = 1715.$$

Since 2005-1715=290, we have to consider only 290 numbers in the sequence with initial digit 8. We have

800000	-	855555	:	252
860000	-	863333	:	35
864000	-	864110	:	3

Thus the required number is 864110.

Solution: II

It is known that the number of ways of choosing r objects from n different types of objects (with repetitions allowed) is $\binom{n+r-1}{r}$. In particular, if we want to write r-digit numbers using n digits allowing for repetitions with the additional condition that the digits appear in non-increasing order, we see that this can be done in $\binom{n+r-1}{r}$ ways.

Now we group the given numbers into different classes and write the number of ways in which each class can be obtained. To keep track we also write the cumulative sums of the number of numbers so obtained. Observe that the numbers themselves are written in ascending order. So we exhaust numbers beginning with 1, then beginning with 2 and so on.

Numbers	Digits used other than the fixed part	n	r	$\binom{n+r-1}{r}$	Cumulative sum
beginning with 1	1,0	2	5	$\binom{6}{5} = 6$	6
2	2,1,0	3	5	$\binom{7}{5} = 21$	27
3	3,2,1,0	4	5	$\binom{8}{5} = 56$	83
4	4,3,2,1,0	5	5	$\binom{9}{5} = 126$	209
5	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	461
6	6,5,4,3,2,1,0	7	5	$\binom{11}{5} = 462$	923
7	7,6,5,4,3,2,1,0	8	5	$\binom{12}{5} = 792$	1715
from 800000 to 855555	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	1967
from 860000 to 863333	3,2,1,0	4	4	$\binom{7}{4} = 35$	2002

The next three 6-digit numbers are 864000, 864100, 864110.

Hence the 2005th number in the sequence is 864110.

5. Let x_1 be a given positive integer. A sequence $\langle x_n \rangle_{n=1}^{\infty} = \langle x_1, x_2, x_3, \cdots \rangle$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

   (a) the sequence has an **even** number;

   (b) the sequence has infinitely many even numbers.

Solution:

(a) Let us assume that there are no even numbers in the sequence. This means that x_{n+1} is obtained from x_n, by adding a nonzero even digit of x_n to x_n, for each $n \geq 1$.
Let E be the left most even digit in x_1 which may be taken in the form

$$x_1 = O_1 O_2 \cdots O_k E D_1 D_2 \cdots D_l$$

where $O_1, O_2, \ldots, O_k$ are odd digits ($k \geq 0$); $D_1, D_2, \ldots, D_{l-1}$ are even or odd; and D_l odd, $l \geq 1$.
Since each time we are adding at least 2 to a term of the sequence to get the next term, at some stage, we will have a term of the form

$$x_r = O_1 O_2 \cdots O_k E 999 \cdots 9F$$

where $F = 3, 5, 7$ or 9. Now we are forced to add E to x_r to get x_{r+1}, as it is the only even digit available. After at most four steps of addition, we see that some next term is of the form

$$x_s = O_1 O_2 \cdots O_k G 000 \cdots M$$

where G replaces E of x_r, $G = E + 1$, $M = 1, 3, 5,$ or 7. But x_s has no nonzero even digit contradicting our assumption. Hence the sequence has some even number as its term.

(b) If there are only finitely many even terms and x_t is the last term, then the sequence $\langle x_n \rangle_{n=t+1}^{\infty} = \langle x_{t+1}, x_{t+2}, \ldots \rangle$ is obtained in a similar manner and hence must have an even term by (a), a contradiction. Thus $\langle x_n \rangle_{n=1}^{\infty}$, has infinitely many even terms.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(x^2 + yf(z)\right) = xf(x) + zf(y) \tag{1}$$

for all x, y, z in $\mathbf{R}$. (Here $\mathbf{R}$ denotes the set of all real numbers.)

Solution: Taking $x = y = 0$ in (1), we get $zf(0) = f(0)$ for all $z \in \mathbf{R}$. Hence we obtain $f(0) = 0$. Taking $y = 0$ in (1), we get

$$f(x^2) = xf(x) \tag{2}$$

Similarly $x = 0$ in (1) gives

$$f(yf(z)) = zf(y) \tag{3}$$

Putting $y = 1$ in (3), we get

$$f(f(z)) = zf(1) \quad \forall\, z \in \mathbf{R} \tag{4}$$

Now using (2) and (4), we obtain

$$f(xf(x)) = f(f(x^2)) = x^2 f(1) \tag{5}$$

Put $y = z = x$ in (3) also given

$$f(xf(x)) = xf(x) \tag{6}$$

Comparing (5) and (6), it follows that $x^2 f(1) = xf(x)$. If $x \neq 0$, then $f(x) = cx$, for some constant c. Since $f(0) = 0$, we have $f(x) = cx$ for $x = 0$ as well. Substituting this in (1), we see that

$$c(x^2 + cyz) = cx^2 + cyz$$

or

$$c^2 yz = cyz \quad \forall\, y, z \in \mathbf{R}.$$

This implies that $c^2 = c$. Hence $c = 0$ or 1. We obtain $f(x) = 0$ for all x or $f(x) = x$ for all x. It is easy to verify that these two are solutions of the given equation.

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