INMO–1998

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1. In a circle C_1 with centre O, let AB be a chord that is not a diameter. Let M be the midpoint of AB. Take a point T on the circle C_2 with OM as diameter. Let the tangent to C_2 at T meet C_1 in P. Show that
$$PA^2 + PB^2 = 4PT^2.$$

2. Let a and b be two positive rational numbers such that $\sqrt[3]{a} + \sqrt[3]{b}$ is also a rational number. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational numbers.

3. Let p, q, r, s be four integers such that s is not divisible by 5. If there is an integer a such that $pa^3 + qa^2 + ra + s$ is divisible by 5, prove that there is an integer b such that $sb^3 + rb^2 + qb + p$ is also divisible by 5.

4. Suppose $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius one unit. If
$$AB \cdot BC \cdot CD \cdot DA \geq 4,$$
prove that $ABCD$ is a square.

5. Suppose a, b, c are three real numbers such that the quadratic equation
$$x^2 - (a + b + c)x + (ab + bc + ca) = 0$$
has roots of the form $\alpha \pm i\beta$ where $\alpha > 0$ and $\beta \neq 0$ are real numbers [here $i = \sqrt{-1}$]. Show that
   (i) the numbers a, b, c are all positive;
   (ii) the numbers $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ form the sides of a triangle.

6. It is desired to choose n integers from the collection of $2n$ integers, namely, $0, 0, 1, 1, 2, 2, \ldots, n-1, n-1$ such that the average (that is, the arithmetic mean) of these n chosen integers is itself an integer and as minimum as possible. Show that this can be done for each positive integer n and find this minimum average for each n.