INMO–1993

Time : 4 hours

Attempt as many questions as you possibly can.

Use of calculating aids not permitted

1. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let O be the circumcenter of triangle APB and H be the orthocenter of triangle CPD. Show that the points H, P, O are collinear.
2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that
$$P(n) \cdot P(n+1) = P(M).$$
3. If a, b, c, d are 4 non-negative real numbers and $a + b + c + d = 1$, show that
$$ab + bc + cd \leq 1/4.$$
4. Let ABC be a triangle in a plane Σ. Find the set of all points P (distinct from A, B, C) in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP have the same radii.
5. Show that there is a natural number n such that $n!$ when written in decimal notation (that is, in base 10) ends exactly in 1993 zeros.
6. Let ABC be triangle right-angled at A and S be its circumcircle. Let S_1 be the circle touching the lines AB and AC and the circle S internally. Further let S_2 be the circle touching the lines AB and AC, and the circle S externally. If r_1 and r_2 be the radii of the circles S_1 and S_2 respectively, show that
$$r_1 \cdot r_2 = 4(\text{ area } \Delta ABC).$$
7. Let $A = \{1, 2, 3, \ldots, 100\}$ and B be a subset of A having 53 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.
8. Let f be a bijective (1-1 and onto) function from $A = \{1, 2, 3 \ldots, n\}$ to itself. Show that there is positive number $M \geq 1$ such that
$$f^M(i) = f(i), \text{ for each } i \text{ in } A.$$
f^M denotes the composite function $\underbrace{f \circ f \circ f \circ \ldots \circ f}_{M \text{ times}}$.
9. Show that there exists a convex hexagon in the plane such that
   (a) all its interior angles are equal,
   (b) all its sides are 1, 2, 3, 4, 5, 6 in some order.