The 35th International Mathematical Olympiad (July 13-14, 1994, Hong Kong)

1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j, $1 \leq i \leq j \leq m$, there exists k, $1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \frac{n+1}{2}.$$

2. ABC is an isosceles triangle with $AB = AC$. Suppose that

1. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
2. Q is an arbitrary point on the segment BC different from B and C;
3. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

- (a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.
- (b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

4. Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

5. Let S be the set of real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;
2. $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

6. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \geq 2$.