Third International Olympiad, 1961

1961/1.

Solve the system of equations:

$$\begin{aligned} x+y+z &= a \\ x^2+y^2+z^2 &= b^2 \\ xy &= z^2 \end{aligned}$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.

1961/2.

Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2+b^2+c^2 \geq 4\sqrt{3}T$. In what case does equality hold?

1961/3.

Solve the equation $\cos^n x - \sin^n x = 1$, where n is a natural number.

1961/4.

Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is ≤ 2 and at least one is ≥ 2.

1961/5.

Construct triangle ABC if $AC = b, AB = c$ and $\angle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90^\circ$. Prove that a solution exists if and only if

$$b \tan \frac{\omega}{2} \leq c < b.$$

In what case does the equality hold?

1961/6.
Consider a plane ε and three non-collinear points A, B, C on the same side of ε; suppose the plane determined by these three points is not parallel to ε. In plane a take three arbitrary points A', B', C'. Let L, M, N be the midpoints of segments AA', BB', CC'; let G be the centroid of triangle LMN. (We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.) What is the locus of point G as A', B', C' range independently over the plane ε?