10th KVS Junior Mathematics Olympiad (JMO) – 2007

M.M. 100 Time : 3 hours

Note : Attempt all questions.

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1. Solve

   $| x-1 | + | x | + | x + 1 | = x + 2$

2. Find the greatest number of four digits which when divided by 3, 5, 7, 9 leaves remainders 1, 3, 5, 7 respectively.

3. A printer numbers the pages of a book starting with 1. He uses 3189 digits in all. How many pages does the book have ?

4. ABCD is a parallelogram. P, Q, R and S are points on sides AB, BC, CD and DA respectively such that AP=DR. If the area of the parallelogram is 16 cm^2, find the area of the quadrilateral PQRS.

5. ABC is a right angle triangle with B = 90^o. M is the mid point of AC and BM = $\sqrt{117}$ cm. Sum of the lengths of sides AB and BC is 30 cm. Find the area of the triangle ABC.

6. Solve :

   $$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}}=\frac{a}{x}$$

7. Without actually calculating, find which is greater :

31^{11} or 17^{14}

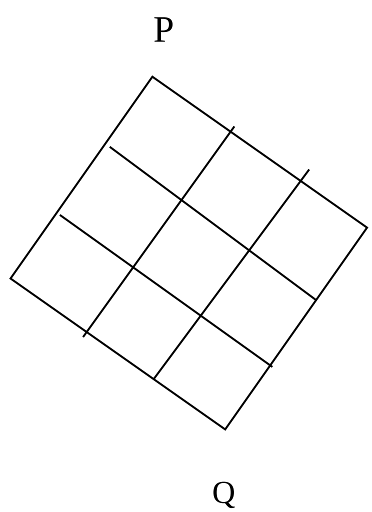
8. Show that there do not exist any distinct natural numbers a, b, c, d such that

   $a^3 + b^3 = c^3 + d^3$ and $a + b = c + d$

9. Find the largest prime factor of :

   $3^{12} + 2^{12} – 2.6^6$

10. If only downward motion along lines is allowed, what is the total number of paths from point P to point Q in the figure below ?



KVJMO – 2007 SOLUTIONS

Q.1. | x-1 | + | x | + | x + 1 | = x + 2

CASE I :

$x > 1$

$| x-1 | + | x | + | x + 1 | = x + 2$

$\Rightarrow \quad \cancel{x} - \cancel{1} + x + x + \cancel{1} = \cancel{x} + 2$

$\Rightarrow \quad 2x = 2$

$\Rightarrow \quad x = 1$

CASE II :

$0 < x < 1$

$\therefore | x - 1| = - x + 1$

$\therefore | x-1 | + | x | + | x + 1 | = x + 2$

$\Rightarrow \quad - \cancel{x} + 1 + \cancel{x} + x + 1 = x + 2$

$\Rightarrow \quad x + 2 = x + 2$

which has trivial solutions.

CASE III :

When, $x < 0$

$| x - 1 | = - x + 1$

$| x | = - x$

$| x + 1 | = - x - 1$

$\therefore | x-1 | + | x | + | x + 1 | = x + 2$

$\Rightarrow \quad - x + \cancel{1} - x - x - \cancel{1} = x + 2$

⇒ $-3x = x + 2$

⇒ $-3x - x = 2$

⇒ $-4x = 2$

⇒ $x = -\frac{1}{2}$

CASE IV :

$x = 0$

$|x-1| + |x| + |x + 1| = x + 2$

⇒ $|0-1| + |0| + |0 + 1| = 0 + 2$

⇒ $|-1| + 0 + |1| = 0 + 2$

⇒ $1 + 1 = 2$

⇒ $2 = 2$

∴ $x = 0$ also satisfies the equation

∴ $x = 1, 0, -\frac{1}{2}$ are the solutions of the equation.

Q2. First,

L.C.M. of 3, 5, 7 and 9 = 315

Now,

The largest four – digit number multiple of 315

= 315×31

= 9765

∴ the required number which leaves remainders of 1, 3, 5, 7 [(3-2), (5-2), (7-2), (9-2)] on division by 3, 5, 7 and 9.

= 9765 – 2

= 9763

Q3. First,

The printer printed 9 single digits numbers (1-9), upto 9

i.e. 9 x 1 = 9 digits.

∴ No. of digits remaining to be printed = 3189 – 9 = 3180

Then, the printer printed 90 2 – digits numbers (10-99), upto 99

i.e. 2 x 90 = 180 digits.

∴ No. of digits remaining to be printed 3180 – 180 = 3000 digits.

Then, the printer printed 900, 3 – digit numbers, (100 – 999) upto 999

i.e. 3 x 900 = 2700 digits.

∴No. of digits remaining to be printed = 3000 – 2700 = 300 digits.

Now, Only 300 more digits are left to be printed.

∴No. of 4-digit numbers which can be printed using 300 digits

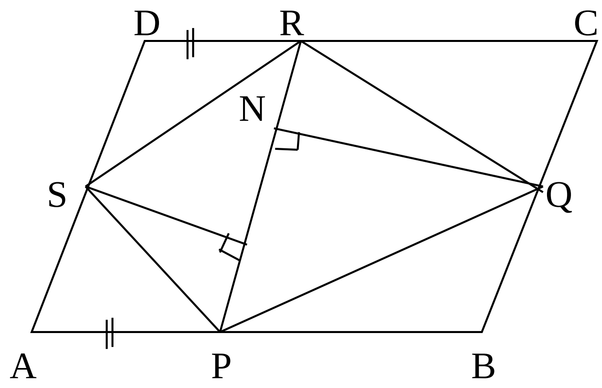
$$= \frac{300}{4} = 75$$

∴75 more numbers can be written after 999.

i.e. (999 + 75) = 1074

∴ The book has 1074 pages.

Q4.



Given :

AP = DR

ar (ABCD) = 16 cm^2

ar (PQ RS) = ?

Construction :

Draw SM⊥PR

and QN⊥PR

Now,

∴ DC||AB

∴ DR||AP

∴ DR||AP and DR=AP

∴ APRD is a parallelogram

∴ ar (APRD) = (Base) x (Height)

= PR x SM

∴ ar (ΔSPR) = ½ x (Base) (Height)

= ½ x PR x SM

= ½ x ar (APRD) (i)

Also,

∵ DC = AB

⇒ ~~DP~~ + RC = ~~AP~~ + PB

⇒ RC = PB

∵ DC || AB

∴ RC || PB

∵ RC || PB and RC = PB

∴ RCBP is a parallelogram

∴ ar (RCBP) = (Base) x (Height)

= PR x QN

∴ (QPR) = ½ x (Base) x (Height)

= ½ x PR x QN

= ½ x ar (RCBP) ….(ii)

Adding (i) and (ii),

ar (ΔSPR) + ar (ΔQPR) = ½ x ar (APRD) + ½ x ar(RCBP)

⇒ ar (PQRS) = ½ (ar (APRD) + ar (RCBP)

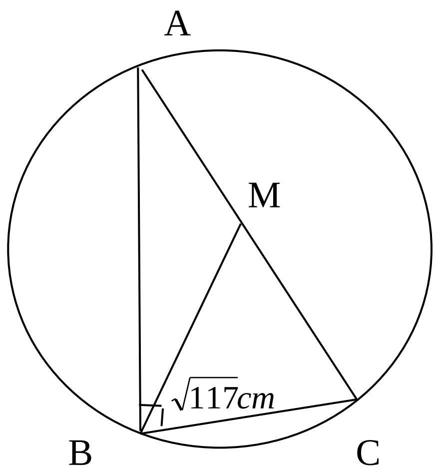
$= \frac{1}{2} \times ar(ABCD)$

$= \frac{1}{2} \times 16$

$= 8\ cm^2$

$\therefore\ ar(PQRS) = 8\ cm^2$

Q5.



Given :

ΔABC is right – angled at B.

M is the mid – point of AC

BM = $\sqrt{117}$ cm

AB + BC = 30 cm

ar (ΔABC) = ?

Const : -

Draw the circumcirle of ΔABC

Now,

$\because$ angles formed on the semi-circle are right angles.

And $\angle$ABC = 90°

$\therefore$ AC must be the diameter,

$\because$ M is the mid-point of AC.

$\therefore$ M is also the centre of the circle.

Hence,

AM = CM = BM = Radius

$\therefore$ AM = CM = BM

$= \sqrt{117}$ cm

$\therefore$ AC = AM + CM

$= \sqrt{117} + = \sqrt{117}$

$= \quad 2\sqrt{117}$

$\therefore$ In ΔABC,

$\angle B = 90^o$

AC = $2\sqrt{117}$ cm

AB + BC = 30 cm

$\therefore$ By Pythagoras theorem, we have,

$AB^2 + BC^2 = AC^2$

$\Rightarrow$ $(AB + BC)^2 - 2.AB.BC = AC^2$

$\Rightarrow$ $(30)^2 - 2.AB.BC = (2\sqrt{117})^2$

$\Rightarrow$ 900 – 2.AB.BC = 4 X 117

$\Rightarrow$ 900 – 4 X 117 = 2.AB.BC

$\Rightarrow$ 2.AB.BC = 900 – 468

$\Rightarrow$ $AB.BC = \frac{432}{2} = 216\ cm^2$

$\therefore$ ar (ΔABC) $= \frac{1}{2}$ x (Base) x (Height)

$= \frac{1}{2}$ x (AB) x (BC)

$= \frac{1}{2}$ x 216

$= 108 \text{ cm}^2$

ar (ΔABC) $= 108 \text{ cm}^2$

Q6. $\frac{\left\{\sqrt{(a+x)}+\sqrt{(a-x)}\right\}}{\left\{\sqrt{(a+x)}+\sqrt{(a-x)}\right\}} = \frac{a}{x}$

$\Rightarrow \quad 1 = \frac{a}{x}$

$\Rightarrow \quad a = x$

CHECK :

L.H.S. $= \frac{\sqrt{(a+a)}+\sqrt{(a-a)}}{\sqrt{(a+a)}+\sqrt{(a-a)}}$

$= \frac{\sqrt{2a}+0}{\sqrt{2a}+0}$

$= 1$

R.H.S.$= \frac{a}{x} = \frac{a}{a} = 1$

$\therefore$ L.H.S. = R.H.S

Verified

Q7. 31^{11} or 17^{14}

$31 < 32$

or $31 < 2^5$

or $31^{11} < (2^5)^{11}$

or $31^{11} < 2^{55}$

$17 > 16$

or $17 > 2^4$

or $17^{14} > (2^4)^{14}$

or $17^{14} > 2^{56}$

$\because$ $31^{11} < 2^{55}$

and $17^{14} > 2^{56}$

and $2^{56} > 2^{55}$

$\therefore$ $17^{14} > 31^{11}$

Q8. $\mathbf{a^3 + b^3 = c^3 + d^3}$ **and** $\mathbf{a + b = c + d}$

$a^3 + b^3 = c^3 + d^3$

$\Rightarrow$ $(a+b)(a^2-ab+b^2) = (c+d)(c^2 - cd + d^2)$

$\Rightarrow$ $a^2 - ab + b^2 = c^2 - cd + d^2$

$\Rightarrow$ $(a+b)^2 - 3ab = (c + d)^2 - 3cd$

$\Rightarrow$ $-3ab = -3cd$

$\Rightarrow$ ab = cd

$\Rightarrow$ $\sqrt{ab} = \sqrt{cd}$

$\therefore$ It does not satisfy the condition.

So, it is not possible for $a^3 + b^3 = c^3 + d^3$ and a + b = c + d,

Simultaneously

$\therefore$ $\frac{a+b}{2} = \frac{c+d}{2}$

$\sqrt{ab} = \sqrt{cd}$

Here, A.M. of a, b is equal to AM of c, d

And G.M. of a, b is equal to G.M. of c, d

This is only possible when a,b and c,d are equal

But, they must be distinct

$\therefore$ It is not possible, if a and b are distinct.

Q9. $\mathbf{3^{12} + 2^{12} - 2.6^6}$

$= (3^6)^2 + (2^6)^2 - 2.(3 \times 2)^6$

$= (3^6)^2 + (2^6)^2 - 2.(3^6) \times (2)^6$

$= (3^6 - 2^6)^2$

$= \left[(3^3)^2 - (2^3)^2\right]^2$

$= \left[(3^3 + 2^3)(3^3 - 2^3)\right]^2$

$= [(3^3+2^3) + (3^3-2^3)]^2$

$= [(3+2)(3^2 - 3 \times 2 + 2^2)(3-2)(3^2 + 3 \times 2 + 2^2)]^2$

$= [(5) \times (9 - 6 + 4)(1) \times (9 + 6 + 4)]^2$

$= [(5) \times (19) \times (7)]^2$

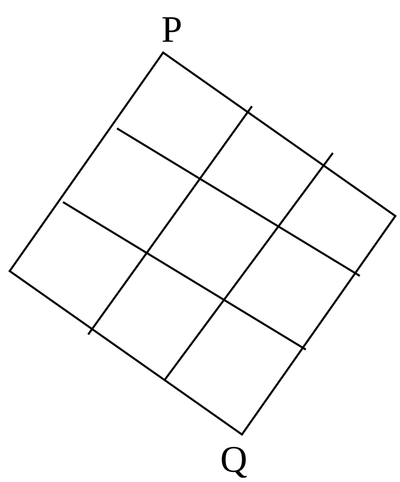
$= [5 \times 19 \times 7]^2$

$= 5^2 \times 19^2 \times 7^2$

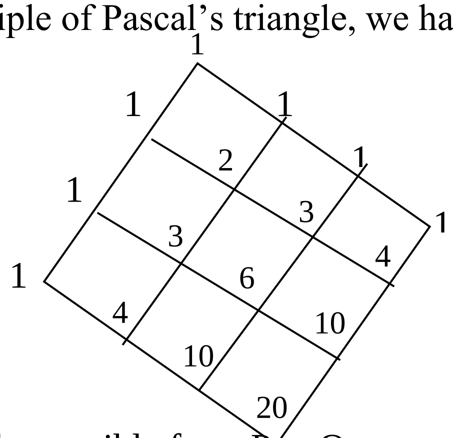
$\therefore$ The largest prime factor of $(3^{12} + 2^{12} - 2.6^6)$

$= 19$

Q10.



By using the principle of Pascal's triangle, we have,



$\therefore$ There are 20 paths possible from P to Q.