Junior Mathematics Olympiad - 2006

Time 3 Hours **M.M.100**

NOTE: Attempt all questions, all questions carry equal marks.

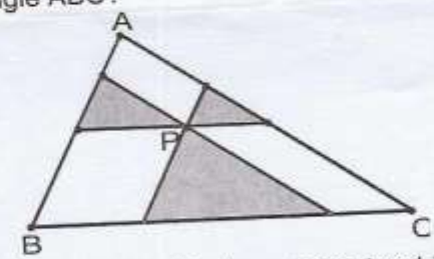
1. a, b, c are three distinct real numbers and there are real numbers x, y such that $a^3 + ax + y = 0$, $b^3 + bx + y = 0$ and $c^3 + cx + y = 0$. Show that $a + b + c = 0$.

2. The triangle ABC has CA = CB. P is a point on the circumcircle between A and B (and on the opposite side of the line AB to C). D is the foot of the perpendicular from C to PB. Show that $PA + PB = 2 \cdot PD$.

3. Given reals x, y with $(x^2 + y^2)/(x^2 - y^2) + (x^2 - y^2)/(x^2 + y^2) = k$, find $(x^8 + y^8)/(x^8 - y^8) + (x^8 - y^8)/(x^8 + y^8)$ in terms of k.

4. In a right triangle ABC right angled at B, a point P is taken on the side AB such that AP = h and PB = b. If BC = d and AC = y such that $h + y = b + d$. Prove that $h = bd/(2b+d)$

5. P is a point inside the triangle ABC. Lines are drawn through P Parallel to the sides of the triangle. The areas of the three resulting triangles with a vertex at P, have areas 4, 9 and 49. What is the area of triangle ABC?



6. A lotus plant in a pool of water is ½ cubit above water level. When propelled by air, the lotus sinks in the pool 2 cubits away from its position. Find the depth of water in the pool.

7. Let C_1 be any point on side AB of a triangle ABC. Join C_1C. The lines through A and B parallel to CC_1 meet BC and AC produced at A_1 and B_1 respectively. Prove that $1/AA_1 + 1/BB_1 = 1/CC_1$

8. The triangle ABC has angle B = 90°. When it is rotated about AB it gives a cone of volume 800π □ . When it is rotated about BC it gives a cone of volume 1920π □ . Find the length AC.

9. A number when divided by 7, 11 and 13 (the prime factor of 1001) successively leave the remainders 6, 10 and 12 respectively. Find the remainder if the number is divided by 1001.

10. Two candles of the same height are lighted together. First one gets burnt up completely in 3 hours while the second in 4 hours. At what point of time, the length of second candle will be double the length of the first candle.

Q.No. 1

$a^3 + ax + y = 0$ ....(i)

$b^3 + bx + y = 0$ ....(ii)

$c^3 + cx + y = 0$ ....(iii)

$\therefore$ (i) = (ii)

$\Rightarrow a^3 + ax + y = b^3 + bx + y$

$\Rightarrow a^3 - b^3 = - ax + bx$

$\Rightarrow (a-b)(a^2+ab+b^2) = -x(a-b)$

$\Rightarrow a^2 + ab + b^2 = -x$ ....(A)

$\therefore$ (ii) = (iii)

$\Rightarrow b^3 + bx + y = c^3 + cx + y$

$\Rightarrow b^3 - c^3 = - bx + cx$

$\Rightarrow (b-c)(b^2+bc+c^2) = - x(b-c)$

$\Longrightarrow b^2 + bc + c^2 = - x$ ....(B)

Also, $\therefore$ (A) = (B)

$\Longrightarrow a^2 + ab + b^2 = b^2 + bc + c^2$

$\Longrightarrow a^2 - c^2 = - ab + bc$

$\Longrightarrow (a+c)(a-c) = - b(a-c)$

$\Longrightarrow a + c = - b$

$\Longrightarrow a + b + c = 0$ Q.E.D.

Q. No. 1: Alternate solution

$a^3 + ax + y = 0$

$\Longrightarrow ax = a^3 - y$

$$\Longrightarrow x = \frac{-a^3 - y}{a} \qquad \text{(i)}$$

$b^3 + bx + y = 0$

$bx = - b^3 - y$

$\Rightarrow x = \dfrac{-b^3 - y}{b}$ (ii)

$\Rightarrow c^3 + cx + y = 0$

$\Rightarrow cx = -c^3 - y$

$\Rightarrow x = \dfrac{-c^3 - y}{c}$ (iii)

Now,

(i) = (ii)

$\Rightarrow \dfrac{-a^3 - y}{a} = \dfrac{-b^3 - y}{b}$

$\Rightarrow -a^3b - by = -ab^3 - ay$

$\Rightarrow ay - by = a^3b - ab^3$

$\Rightarrow y(a-b) = ab(a^2 - b^2)$

$\Rightarrow y = ab(a+b)$

(ii) = (iii) $\Rightarrow y = bc(b+c)$

(i) = (iii) $\Rightarrow \underline{y = ac(a + c)}$

$$\therefore\ 3y = ab\,(a + b) + bc\,(b + c) + ac\,(a + c)$$

Now

$$a^3 + ax + y = b^3 + bx + y$$

$$\Rightarrow \quad a^3 - b^3 = -\,ax + bx$$

$$\Rightarrow \quad (a-b)\ (a^2 + ab + b^2) = -x\,(a-b)$$

$$\Rightarrow \quad -x = a^2 + ab + b^2 \ \ldots\ldots\ldots\ldots\ldots\ (iv)$$

$$a^3 + ax + y = c^3 + cx + y$$

$$\Rightarrow \quad a^3 - c^3 = -\,ax + cx$$

$$\Rightarrow \quad (a-c)\ (a^2 + ac + c^2) = -\,x\,(a-c)$$

$$\Rightarrow \quad -x = a^2 + ac + c^2 \ \ldots\ldots\ldots\ldots\ (v)$$

In the similar fashion,

$$-x = b^2 + bc + c^2 \ \ldots\ldots\ldots\ldots\ (vi)$$

Adding, (iv) + (v) + (vi)

$$\Rightarrow \quad -3x = 2a^2 + 2b^2 + 2c^2 + ab + bc + ca$$

$a^3 + ax + y = 0$

$b^3 + bx + y = 0$

$c^3 + cx + y = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 + ax + bx + cx + 3y = 0$

$\Rightarrow \quad a^3+b^3+c^3+x(a+b+c) + ab(a+b) + bc(b+c) + ac(a+c) = 0$

$$\rightarrow a^3 + b^3 + c^3 - \frac{1}{3}[2a^2 + 2b^2 + 2c^2 + ab + bc + ca](a+b+c) + ab(a+b) + \ldots + ac(a+c) = 0$$

$$\rightarrow a^3 + b^3 + c^3 - (+\frac{2}{3}a^3 + \frac{2}{3}ab^2 + \frac{2}{3}ac^2 + \frac{1}{3}a^2b + \frac{1}{3}abc + \frac{1}{3}a^2c$$

$$+\frac{2}{3}a^2b + \frac{2}{3}b^3 + \frac{2}{3}bc^2 + \frac{1}{3}ab^2 + \frac{1}{3}b^2c + \frac{1}{3}abc$$

$$+\frac{2}{3}a^2c + \frac{2}{3}b^2c + \frac{2}{3}c^3 + \frac{1}{3}abc + \frac{1}{3}bc^2 + \frac{1}{3}ac^2)$$

$+ a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2 = 0$

$$\Rightarrow \quad a^3 + b^3 + c^3 - \frac{2}{3}a^3 - \frac{2}{3}ab^2 - \frac{2}{3}ac^2 - \frac{1}{3}a^2b - \frac{1}{3}abc - \frac{1}{3}a^2c$$

$$-\frac{2}{3}a^2b - \frac{2}{3}b^3 - \frac{2}{3}bc^2 - \frac{1}{3}ab^2 - \frac{1}{3}b^2c - \frac{1}{3}abc$$

$$-\frac{2}{3}a^2c-\frac{2}{3}b^2c-\frac{2}{3}c^3-\frac{1}{3}abc-\frac{1}{3}bc^2-\frac{1}{3}ac^2$$

$$+a^2b+ab^2+b^2c+bc^2+a^2c+ac^2=0$$

$$\Rightarrow \frac{1}{3}a^3+\frac{1}{3}b^3+\frac{1}{3}c^3-abc=0$$

$\Rightarrow$ $a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow$ $\{a^3 + (b+c)^3\} - 3bc\,(b+c) - 3abc = 0$

$\Rightarrow$ $(a+ b+c)\,\{a^2 - a(b+c) + (b+c)^2\} = 0$

$\Rightarrow (a+b+c)\,(a^2-a\,(b+c) + (b+c)^2 - 3bc\,(a+b+c) = 0$

$\Rightarrow (a+b+c)\,[a^2-a(b+c) + (b+c)^2 - 3bc] = 0$

$\Rightarrow (a+b+c)\,[a^2-ab-ac+b^2+c^2+2bc-3bc] = 0$

$\Rightarrow (a+b+c)\,[a^2-ab+b^2-bc+c^2-ca] = 0$

$\Rightarrow \frac{1}{2}(a+b+c)\,[2a^2 - 2ab + 2b^2 - 2bc + 2c^2 - 2ca] = 0$

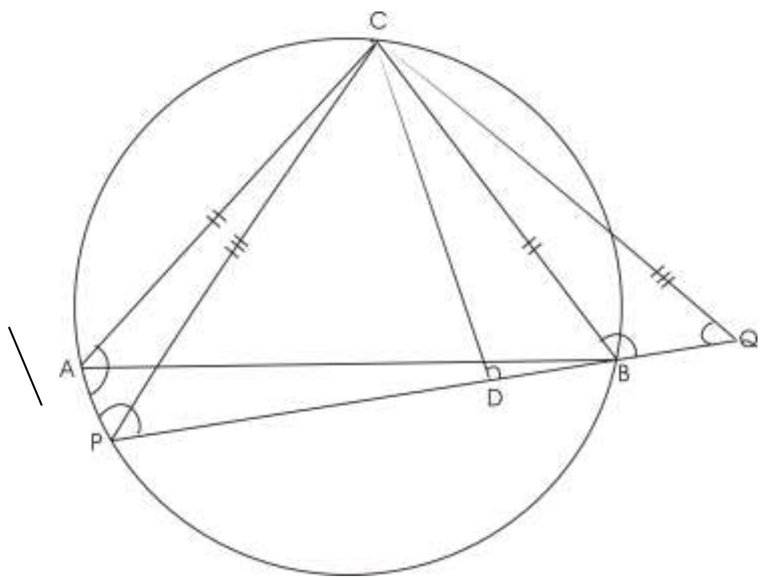
$$\Rightarrow \frac{1}{2}(a+b+c)\left[a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2\right]=0$$

$$\Rightarrow \frac{1}{2}(a+b+c)\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}=0$$

a, b and c are real so. $(a-b)^2$, $(b-c)^2$ and $(c-a)^2$ is positive. Hence $(a-b)^2 + (b-c)^2 + (c-a)^2 > o$,

so, a+b+c = 0 QED

Q2. In the figure



Let us extend PD to DQ

Such that PD = DQ

and CA = CB from question

as CD is ⊥PDQ and PD = DQ

so PC = QC

In ΔACP and ΔBCQ

AC = CB

CP = CQ

∠CAP and ∠CBP are supplementary

[as CAPB is cyclic]

∠APC = ∠CQB, therefore

∠CAP = ∠CBQ

chord BC has angled CAB and CPQ so

so, ∠CAB = ∠CPQ

and so, ∠CPQ = ∠CQP

So, ΔACP and ΔBCQ are congruent

So, AP = BQ

So, 2PD= PQ

= PB + BQ

= PB + AP

Q.3. $\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}$

= $\frac{(x^2+y^2)^2+(x^2-y^2)^2}{(x^2-y^2)(x^2+y^2)}$

= $\frac{x^4+2x^2y^2+y^4+x^4-2x^2y^2+y^4}{x^4-y^4}$

= $\frac{2(x^4+y^4)}{x^4-y^4}$

$\therefore\ \frac{x^4+y^4}{x^4-y^4}=\frac{k}{2}$

By Componendo and Dividendo,

$\Rightarrow\frac{x^4+y^4+x^4-y^4}{x^4+y^4-x^4+y^4}=\frac{k+2}{k-2}$

$\Rightarrow\frac{x^4}{y^4}=\frac{k+2}{k-2}$

Now,

$$= \frac{(x^8 + y^8)/y^8}{(x^8 + y^8)/y^8} + \frac{(x^8 - y^8)/y^8}{(x^8 + y^8)/y^8}$$

$$= \frac{\frac{x^8}{y^8} + 1}{\frac{x^8}{y^8} - 1} + \frac{\frac{x^8}{y^8} - 1}{\frac{x^8}{y^8} + 1}$$

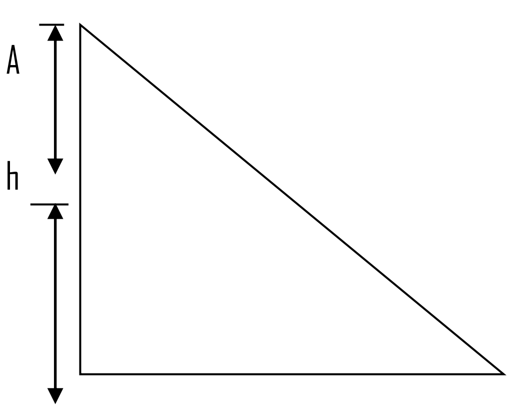
$$= \frac{\left(\frac{k+2}{k-2}\right)^2 + 1}{\left(\frac{k+2}{k-2}\right)^2 - 1} + \frac{\left(\frac{k+2}{k-2}\right)^2 - 1}{\left(\frac{k+2}{k-2}\right)^2 + 1}$$

$$= \frac{k^2 + 4k + 4 + k^2 - 4k + 4}{k^2 + 4k + 4 - k^2 + 4k - 4} + \frac{k^2 + 4k + 4 - k^2 + 4k + 4}{k^2 + 4k + 4 + k^2 - 4k + 4}$$

$$= \frac{2(k^2 + 4)}{8k} + \frac{8k}{2(k^2 + 4)}$$

$$= \frac{(k^2 + 4)}{4k} + \frac{4k}{k^2 + 4}$$

4.



P

b

B ***d*** ***C***

Given, $h + y = b + d$

By Pythagoras theorem,

$$(h+b)^2 + d^2 = y^2$$

$\Rightarrow (h+b)^2 + d^2 = (b+d-h)^2$

$\Rightarrow h^2 + b^2 + 2bh+d^2 = b^2 + d^2 + h^2 + 2bd - 2dh - 2bh$

$\Rightarrow 2bh + 2dh + 2bh = 2bd$

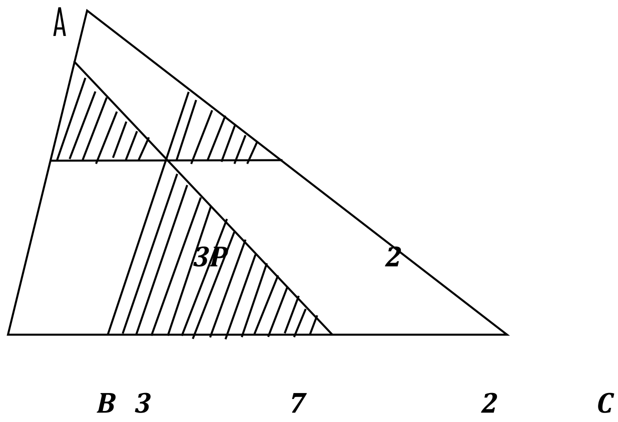
$\Rightarrow 2bh + dh = bd$

$\Rightarrow h(2b + d) = bd$

$$\Rightarrow h = \frac{bd}{2b+d}$$

Q.E.D.

Q5.



Here, all the four triangles are similar to each other.

The areas of the shaded triangles are 4, 9 and 49.

We know that,

The ratio of the areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

∴ The basis of the similar Δs the ratio of sides will be 2, 3 and 7. As shown in the above diagram.

∴ BC = 3 + 7 + 2

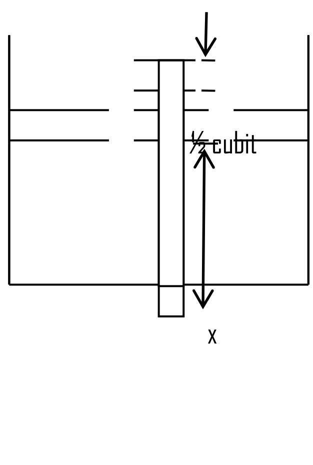
=12

∴ Area of ABC is the square of BC.

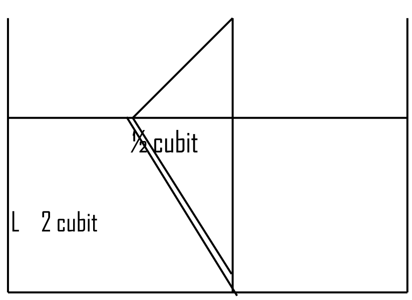
$= (12)^2$

$= 144$ sq units

Q6.



INITIAL POSITION



x+ ½ x

FINAL POSITION

$\therefore$ Let, depth of water in pool = x

Now, by Pythagoras theorem,

$$\left(x+\frac{1}{2}\right)^2 = (2)^2 + x^2$$

$$\Rightarrow x^2 + x + \frac{1}{4} = x^2 + 4$$

$$\Rightarrow x = 4 - \frac{1}{4}$$

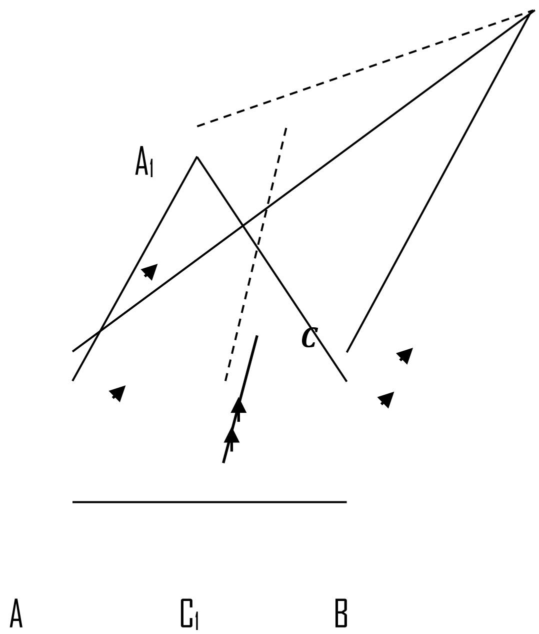
$$\Rightarrow x = \frac{16-1}{4}$$

$$\Rightarrow \frac{15}{4}$$

$\Rightarrow \quad 3\frac{3}{4}$ cubit

$\therefore$ Depth of water is pool = $3\frac{3}{4}$ cubit

7.



To prove :

$$\frac{1}{AA_1}+\frac{1}{BB_1}=\frac{1}{CC_1}$$

Proof :

AA_1, BB_1 and CC_1 are parallel line segments and hence

$$\frac{CC_1}{A_1A}=\frac{C_1B}{AB} \qquad (1)$$

Also $\frac{CC_1}{B_1B} = \frac{AC_1}{AB}$ (2)

Adding (1) and (2), we have

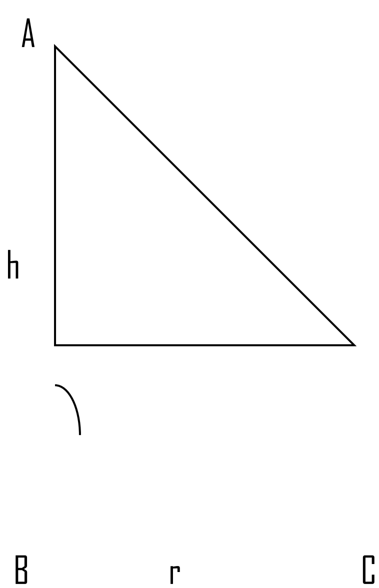
$$\frac{CC_1}{A_1A} + \frac{CC_1}{B_1B} = \frac{C_1B + AC_1}{AB}$$

$$\Rightarrow \frac{AB}{AB} = 1$$

$$\Rightarrow CC_1\left(\frac{1}{A_1A} + \frac{1}{B_1B}\right) = 1$$

$$\Rightarrow \frac{1}{AA_1} + \frac{1}{BB_1} = \frac{1}{CC_1}$$

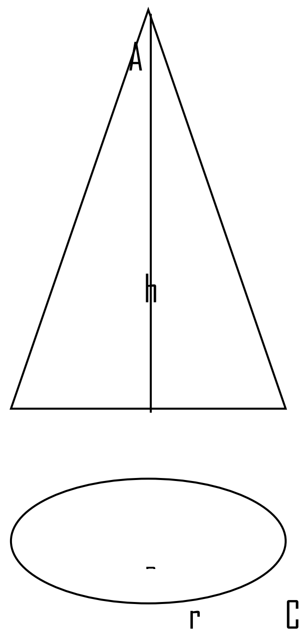
8.



Let, AB = h

And BC = r

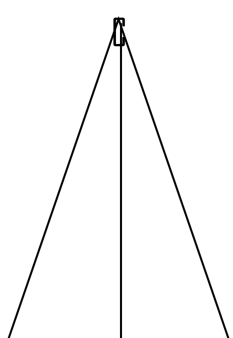
1st case :

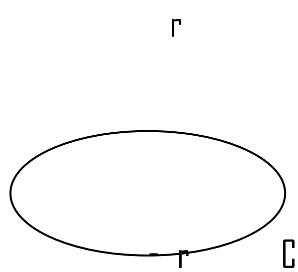


$\therefore$ volume = $\frac{1}{3}\pi r^2 h$

$\Rightarrow 800\pi = \frac{1}{3}\pi r^2 h$ (i)

2nd Case :





$\therefore$ volume = $\frac{1}{3}\pi h^2 r$

$\Rightarrow 1920\,\pi = \frac{1}{3}\pi\, r\, h^2$ (ii)

Now, $\frac{(i)}{(ii)}$

$$\Rightarrow \frac{\frac{1}{3} r^2 h}{\frac{1}{3} r h^2} = \frac{800\pi}{1920\pi}$$

$$\Rightarrow \frac{r}{h} = \frac{80}{192}$$

$$\Rightarrow \frac{r}{h} = \frac{5}{12}$$

Let, r = 5x

h = 12 x

In (i), putting r = 5x and h = 12x,

$\frac{1}{3}r^2h = 800$

$\Rightarrow \frac{1}{3}\ X\ 25x^2\ X\ 12x = 800$

$\Rightarrow x^3 = \frac{800}{25x4}$

$\Rightarrow x^3 = 3\sqrt{8}$

= 2

$\therefore$ h = 12 x 2 = 24

= 24

$\therefore$ r = 5 x 2 = 10

$\therefore$ Hypotenuse AC = $\sqrt{h^2 + r^2}$

= $\sqrt{(24)^2 + (10)^2}$

= $\sqrt{1152 + 200}$

= $\sqrt{1352}$

= 26 units

$\therefore$ AC = 26 units.

9. L.C.M. of 7, 11 and 13 = 1001

So, the required no. can be 1001-1 or 2002 – 1 and so on

= 1000 = 2001

$\therefore$ The remainder when it is divided by 1001 is always = 1000

10. **<u>Ist candle :</u>**

Let, the length = 1 units

$\therefore$ After 1 hour only 2/3 remain (1/3 burnt)

$\therefore$ After 1 minute amount burnt = $\frac{1}{3} \text{x} \frac{1}{60}$

Amount remaining after 1 min = $1-\frac{1}{180}$

= $\frac{179}{180}$

Amount remaining after 2 min = $1-\frac{2}{180}$

= $\frac{178}{180}$

$\therefore$ Amount remaining x min = $1-\frac{x}{180}$

2nd Candle

Let, the length = 1 units

$\therefore$ After 1 hour amount burnt = $\frac{1}{4}$

$\therefore$ After 1 minute amount burnt = $\frac{1}{4} x \frac{1}{60}$

Amount remaining after 1 min = $1-\frac{1}{240}$

$= \frac{239}{240}$

Amount remaining after 2 min = $1-\frac{2}{240}$

$= \frac{238}{240}$

$\therefore$ Amount remaining after x min = $1-\frac{x}{240}$

$\therefore$ Now, as per the given condition, we have,

$$2\left(1-\frac{x}{180}\right)=1-\frac{x}{240}$$

$$\Rightarrow 2-\frac{x}{90}=1-\frac{x}{240}$$

$\Rightarrow \frac{180-x}{90} = \frac{240-x}{240}$

$\Rightarrow 8(180-x) = 3(240-x)$

$\Rightarrow 1440-8x = 720 - 3x$

$\Rightarrow 8x-3x = 1440 - 720$

$\Rightarrow 5x = 720$

$\Rightarrow x = \frac{720}{5}$

= 144 minutes

$\therefore$ Ans : 144 minutes.

After 144 minutes the second candle will be double of the 1st candle.