KVS Junior Mathematics Olympiad (JMO) – 2004

M.M. 100 Time : 3 hours

Note : (i) Attempt all questions. Each question carries ten marks.

(ii) Please check that there are two printed pages and ten questions in the question paper

1. Fill in the blanks :

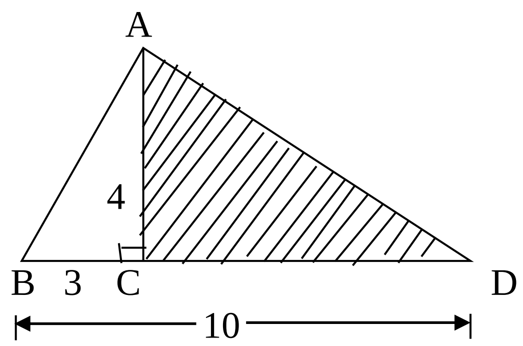
(a) The number of hours from 8 p.m. Tuesday until 5 am Friday of the same week is …………….

(b) If $3^{x-2} = 81$, then x equals …..

(c) In a school the ratio of boys to girls is 3:5 and the ratio of girls to teachers is 6:1. The ratio of boys of teachers is ……..

(d) If $7n + 9 > 100$ and n is an integer, the smallest possible value of n is …….

(e) In the diagram, AC = 4, BC = 3, and BD = 10. The area of the shaded triangle is ……..



2. (a) Find the number of positive integers less than or equal to 300 that are multiples of 3 or 5, but are not multiples of 10 or l5.

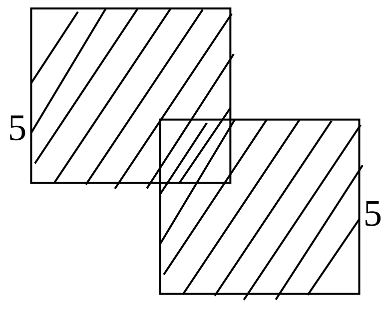
(b) The product of the digits of each of the three-digit numbers 138, 262, and 432 is 24. Write down all three-digit numbers having 24 as the product of the digits.

3. (a) Solve : $x^2 + xy + y^2 = 19$

$$x^2 - xy + y^2 = 49$$

(b) The quadratic polynomials $p(x) = a(x-3)^2 + bx + 1$ and $q(x) = 2x^2 + c(x-2) + 13$ are equal for all values of x. Find the values of a, b and c.

4. (a) Two squares, each with side length 5 cm, overlap as shown. The shape of their overlap is a square, which has an area of $4 cm^2$. Find the perimeter, in centimeters of the shaded figure.

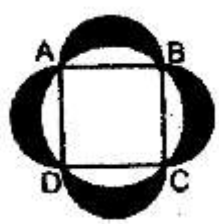


(b) A rectangle is divided into four smaller rectangles. The areas of three of these rectangles are 6, 15 and 25, as shown. Find the area of the shaded rectangle.

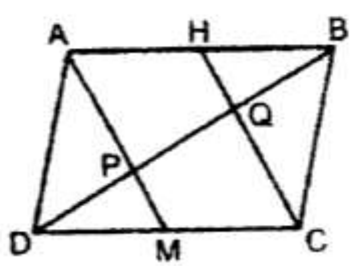
6	15
	25

2004_Q_part_2

5. (a) A square ABCD is inscribed in a circle of unit radius. Semi-circles are described on each side as a diameter. Find the area of the region bounded by the four semi-circles and the circle.



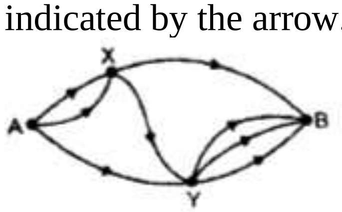
(b) In a parallelogram ABCD, H is the mid-point of AB and M is the mid-point of CD. Show that AM and CH divide the diagonal DB in three equal parts.



2004_part_3

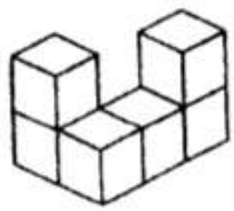
6. A two-digit number has the property that the square of its tens digit plus ten times its units digit is equal to the square of its units digit plus ten times its ten digit. Find all two digit numbers which have this property, and are prime numbers.

7. In the diagram, it is possible to travel only along an edge in the direction indicated by the arrow. How many different routes from A to B are there in all ?

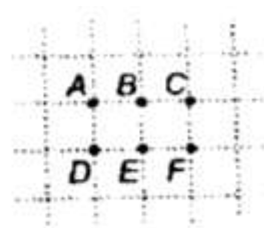


Part_4

8. The object shown in the diagram is made by gluing together the adjacent faces of six wooden cubes, each having edges of length 2 cm. Find the total surface area of the object in square centimeters.



9. Six points A, B, C, D, E and F are placed on a square grid, as shown. How many triangles that are not right-angled can be drawn by using 3 of these 6 points as vertices.



10. A distance of 200 km is to be covered by car in less than 10 hours. Yash does it in two parts. He first drives for 150 km at an average speed of 36 km/hr, without stopping. After taking rest for 30 minutes, he starts again and covers the remaining distance non-stop. His average for the entire journey (including the period of rest) exceeds that for the second part by 5km/hr. Find the speed at which he covers the second part.

KV JMO 2004 SOLUTION

Q1.(a) 57 hrs.

(b) $3^{x-2} = 81$

$\Rightarrow 3^{x-2} = 3 \times 3 \times 3 \times 3$

$\Rightarrow 3^{x-2} = (3)^4$

$\Rightarrow x-2 = 4$

$\Rightarrow x = 6$

$\therefore$ x = 6 Ans

(c) We have,

$$\frac{\text{Boys}}{\text{Girls}} = \frac{3}{5}$$

$$\Rightarrow \frac{B}{G} = \frac{3}{5}$$

$$\Rightarrow \frac{\text{Girls}}{\text{Teachers}} = \frac{6}{1}$$

$$\Rightarrow \frac{G}{T} = \frac{6}{1}$$

$\Rightarrow G = 6T$

$$\therefore \frac{B}{6T} = \frac{3}{5}$$

$$\Rightarrow \frac{B}{T} = \frac{6x3}{5} = \frac{18}{5}$$

∴ The required ratio = 18:5 .

(d) $7n + 9 > 100$

$\Rightarrow 7n > 100 - 9$

$\Rightarrow 7n > 91$

For n = 13

7 x 13 = 91

For n = 14

7 x 14 = 98 > 91

∴ n = 14 Ans

(e) Area of the shaded region = ar (ΔACD)

$\Rightarrow \frac{1}{2}$ x CD x AC

$\Rightarrow \frac{1}{2}$ x (BD-BC) x AC

$\Rightarrow \frac{1}{2}$ x (10-3) x 4

$\Rightarrow$ 7 x 2

$\Rightarrow$ 14 cm^2

Q2. (a) They are :

3 x 1	5

3 x 2

3 x 3

3 x 4

3 x 6

|

|

3 x 98

3 x 99

i.e. 80 numbers

5 x 5

5 x 7

5 x 11

5 x 13

5 x 17

5 x 19

5 x 23

5 x 25

5 x 29

5 x 31

5 x 41

5 x 43

5 x 47

5 x 49

5 x 53

5 x 59

i.e. 18 numbers

which accounts for a total of 98 numbers.

(b) $\because$ 24 = 2 x 2 x 2 x 3

$\therefore$ 24 can be written as

4 x 2 x 3, 8 x 1 x 3, 4 x 6 x 1, 2 x 2 x 6

$\therefore$ All the numbers are :

with digits 4,2 and 3 $\Rightarrow$ 423, 234, 243, 432, 342, 324

with digit 8, 1 and 3 $\Rightarrow$ 138, 183, 318, 381, 813, 831

with digit 4, 6 and 1 $\Rightarrow$ 164, 146, 461, 416, 641, 614

with digit 2, 2 and 6 $\Rightarrow$ 226, 262, 622

3. (a) $x^2 + xy + y^2 = 19$

$\Rightarrow xy = 19 - x^2 - y^2$

$\therefore x^2 - xy + y^2 = 49$

$\Rightarrow x^2 - (19-x^2-y^2) + y^2 = 49$

$\Rightarrow 2x^2 + 2y^2 = 49 + 19$

$\Rightarrow x^2 + y^2 = \frac{68}{2}$

$\Rightarrow x^2 + y^2 = 34$

$\therefore x^2 + xy + y^2 = 19$

$\Rightarrow 34 + xy = 19$

$\Rightarrow xy = 19 - 34$

$= -15$

$\therefore (x+ y)^2 = x^2 + y^2 + 2xy$

$= 34 + 2x(-15)$

$= 34 - 30$

$= 4$

$\Rightarrow x + y = 4$

$(x-y)^2 = x^2 + y^2 - 2xy$

$= 34 - 2(-15)$

$= 34 + 30$

$= 64$

$\Rightarrow x - y = 8$

$x + y = 4$

$x - y = 8$

$2x = 12$

$\Rightarrow x = 6$

$\therefore 6+y = 4$

$\Rightarrow y = -2$

$\therefore x = 6$

$\therefore y = -2$

3(b) $p(x) = a(x-3)^2 + bx + 1$

$q(x) = 2x^2 + c(x-2) + 13$

$p(3) = a(3-3)2 + 3b + 1$

$= 3b + 1$

q(3) $= 2 \times 3^2 + c(3-2) + 13$

$= 18 + c + 13$

$= 31 + c$

$\therefore\ 3b + 1 = 31 + c$

$\Rightarrow 3b - c = 30$ (i)

p(2) $= a\ (2-3)^2 + 2b + 1$

$= a+2b + 1$

q(2) $= 2 \times 2^2 + c\ (2-2) + 13$

$= 4 + 13$

$=17$

$\therefore\ a + 2b + 1 = 17$

$\Rightarrow a + 2b = 16$ (ii)

p(o) $= a\ (0-3)^2 + 6(0) + 1$

$= 9a +1$

q(o) $= 2x0^2 + 6(0-2) + 13$

$= -2c + 13$

$\therefore\ 9a + 1 = -2c + 13$

$\Rightarrow 9a + 2c = 12$ (iii)

We have the following three eqn :

$3b - c = 30$

$a + 2b = 16$

$9a + 2c = 12$

$$\begin{array}{rl} & 6b - 2c = 60 \\ + & 9a + 2c = 12 \\ \hline & 9a + 6b = 72 \end{array}$$

$\Rightarrow 3a + 2b = 24$

$$\begin{array}{l} 3a + 2b = 24 \\ a + 2b = 16 \\ \hline 2a = 8 \end{array}$$

$\Rightarrow a = 4$

$\therefore\ 4 + 2b = 16$

$\Rightarrow 2b = 16 - 4$

$\Rightarrow b = 16 - 4$

$\Rightarrow b = \dfrac{12}{2} = 6$

$9 \times 4 + 2c = 12$

$\Rightarrow 2c = 12 - 36$

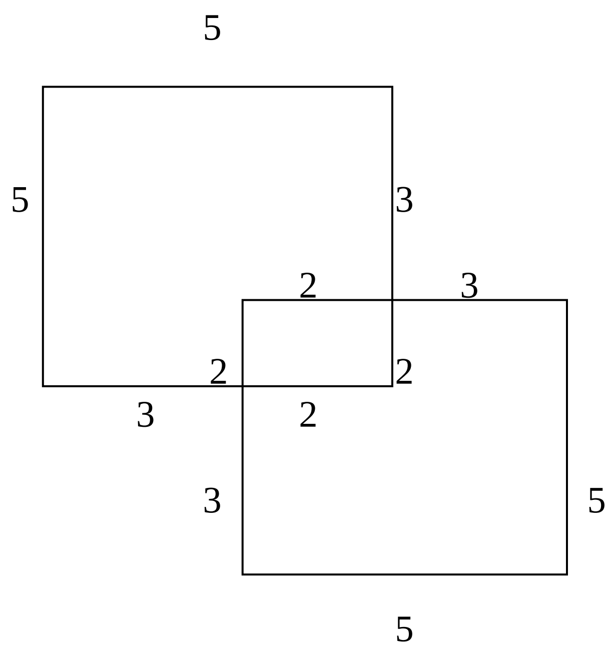
$\Rightarrow c = -\dfrac{24}{2} = -12$

$\therefore$ a = 4

b = 6

c = -12

4. (a)



∵ The overlapping region has an area of 4 cm^2.

∴ It's sides must be 2 cm each.

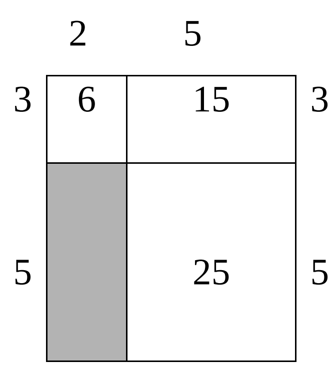
∴ We get the sides as shown in the above figure.

∴ Perimeter of the above figure = 5 + 5 + 3 + 3

+ 5 + 5 + 3 + 3

= 32 cm

(b)

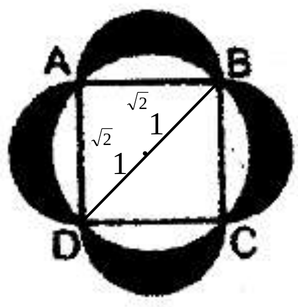


2 5

By the area given, we find the sides of the rectangles as shown in the above figure.

$\therefore$ Area of the shaded region = 5 x 2 = 10 sq unit

Q5.(a)



The total area of the above figure

$= (\sqrt{2})^2 + 4 \text{ x } \frac{\pi r^2}{2}$

$= 2 + 4 \text{ x } \pi \text{ x } \left(\frac{1}{\sqrt{2}}\right)^2 \text{ x } \frac{1}{2}$

$= 2 + 2 \pi \text{ x } \text{ x } \frac{1}{2}$

$= 2 + \pi$

Area of the circle inscribing the square = πr^2

$= \pi \text{ x } 1^2$

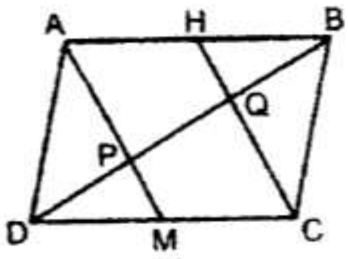
$= \pi$

$\therefore$ Area of the shaded figure = ar (total figure) - ar (circle)

$= 2 + \pi - \pi$

= 2 sq. units

5. (b)



We have,

A H = MC

and AH||MC

$\therefore$ AHCH is a parallelogram

$\therefore$ AM||HC

In ΔABP,

$\because$ AP||HQ

$\therefore \frac{AM}{BH} = \frac{PQ}{QB}$

$\Rightarrow 1 = \frac{PQ}{QB}\left[\because AH = HB = \frac{1}{2}AB\right]$

$\Rightarrow$ QB = PQ (i)

In ΔDQC,

$\because$ PM||QC

$\therefore \frac{DM}{MC} = \frac{DP}{PQ}$

$\Rightarrow 1 = \frac{DP}{PQ}\left[\because DM = MC = \frac{1}{2}CD\right]$

$\Rightarrow PQ = DP$ (ii)

From (i) and (ii)

DP = PQ = QB i.e. BD is trisected.

Q6. Let, unit's digit = y

Ten's digit = x

$\therefore x^2 + 10y = y^2 + 10x$

$\Rightarrow x^2 - 10y = y^2 - 10y$

$\Rightarrow x^2 - y^2 = 10x - 10y$

$\Rightarrow (x + y)(x - y) = 10(x - y)$

$\Rightarrow x + y = 10$

we get the following table of solutions,

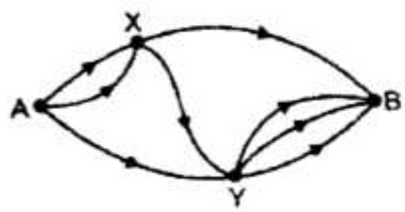
X	1	2	3	4	5	6	7	8	9
Y	9	8	7	6	5	4	3	2	1
10x + y	19	28	37	46	55	64	73	82	91

Here,

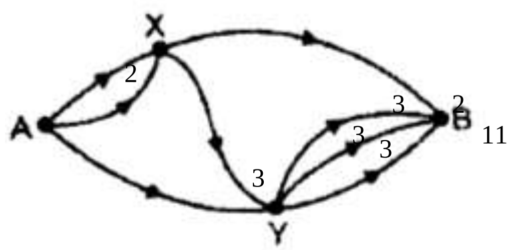
19, 37 and 73 are prime numbers.

∴ The required prime numbers are 19, 37 and 73

Q7.

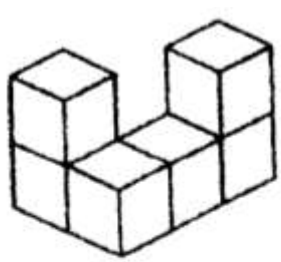


By using the principle of Pascal's triangle, we have



∴ There are 11 ways to go from A to B.

Q8.



From the Top we can see 4 surfaces

From Left, we can see 4 surfaces.

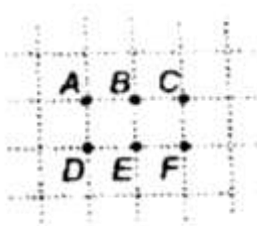
From Right, we can see 5 surfaces.

$\therefore$ It's total S.A. $= 2\,(4+4+5) \times a^2$

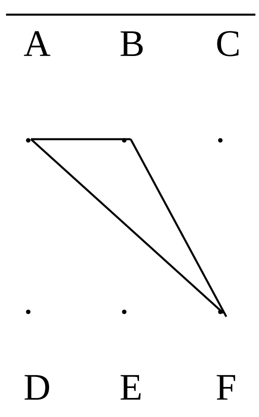
$= 2 \times 13 \times 2^2$

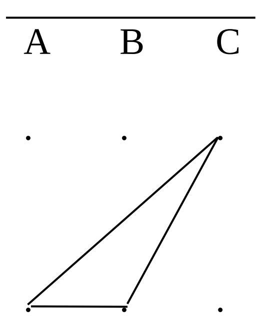
$= 104$ sq. units.

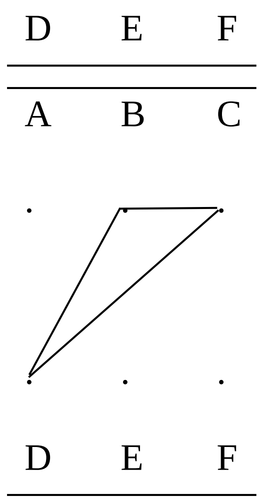
Q9.

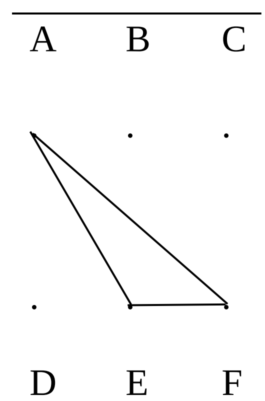


The various possible triangles are :-









So, there are only 4 possible triangles.

Q.10 **<u>IST PART</u>** :

Distance = 150 km

Speed = 36 km/h

$\therefore$ Time = $\frac{\text{Distance}}{\text{Speed}}$

$= \frac{150}{36}$

$= \frac{25}{6}$ hrs

RESTING PART :

Distance = 0 km

Time = 30 min

$= \frac{1}{2}$ hrs.

IIND PART :

Distance = 50 km

Let, time = x hr.

$\therefore$ Speed = $\frac{50}{x}$

$\therefore$ Average speed for the IInd Part = $\frac{50}{x}$ km/h

Average speed for the entire journey = $\frac{\text{Total Distance}}{\text{Total Time}}$

$$= \frac{150+0+50}{\frac{25}{6}+\frac{1}{2}+x}$$

$$= \frac{200}{\frac{25+3+6x}{6}}$$

$$= \frac{1200}{28+6x} \text{ km/h}$$

$= \dfrac{600}{14+3x}\,\text{km/h}$

$\therefore$ According to the question,

$\Rightarrow \quad \dfrac{600}{14+3x} - \dfrac{50}{x} = 5$

$\Rightarrow \quad \dfrac{600x - 50(14+3x)}{x(14+3x)} = 5$

$\Rightarrow \quad 600\,x - 50\,(14+3x) = 5\,(14x + 3x^2)$

$\Rightarrow \quad 120\,x - 140 - 30x = 14x + 3x^2$

$\Rightarrow \quad 90\,x - 14x - 3x^2 = 140$

$\Rightarrow \quad -3x^2 - 76x + 140 = 0$

$\therefore\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \dfrac{-(-76) \pm \sqrt{(-76)^2 - 4(3)(140)}}{2x3}$

$= \dfrac{76 \pm \sqrt{5776 - 1680}}{6}$

$= \dfrac{76 \pm \sqrt{4096}}{6}$

$= \dfrac{76 \pm 64}{6}$

$\therefore\ x = \dfrac{76+64}{6}$ or $\dfrac{76-64}{6}$

$= \frac{140}{6} \text{ or } \frac{12}{6}$

$= \frac{70}{3}$ or 2

For, x = $\frac{70}{3}$

Total time = $\frac{25}{6} + \frac{1}{2} + \frac{70}{3}$

$= \frac{25 + 3 + 140}{6}$

$= \frac{168}{6} = 28$ hrs.

which is impossible

For, x = 2,

Total time = $\frac{25}{6} + \frac{1}{2} + 2$

$= \frac{25 + 3 + 12}{6}$

$= \frac{40}{6}$

$= \frac{20}{3}$

$= 6\frac{2}{3}$ hrs.

which is possible.

$\therefore$ x = 2

$\therefore$ Speed for the IInd Part = $\frac{50}{x}$

$\frac{50}{2}$ = 25 km/h

$\therefore$ Speed for the IInd Part = 25 km/h

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