KVS Junior Mathematics Olympiad (JMO) – 2003

M.M. 100 Time : 3 hours

Note : *(i)* *Please check that there are two printed pages and 10 questions in the question paper.*

(ii) *All questions carry equal marks.*

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1. Fill in the blanks

(a) The digits of the number 2978 are arranged first in descending order and then in ascending order. The difference between the resulting two numbers is ……………

(b) Yash is riding his bicycle at a constant speed of 12 kilometers per hour. The number of metres he travels each minute is ………..

(c) The square root of 35 x 65 x 91 is ………..

(d) The number 81 is 15% of …………..

(e) A train leaves New Delhi at 9.45 am and reaches Agra at 12.58 pm. The time taken in the journey, in minutes, is …………..

2. (a) Find the largest prime factor of 203203.

(b) Find the last two (tens' and units') digits of $(2003)^{2003}$.

3. (a) Find the number of perfect cubes between 1 and 1000009 which are exactly divisible by 9.

(b) If $x = 5 + 2\sqrt{6}$, find the value of

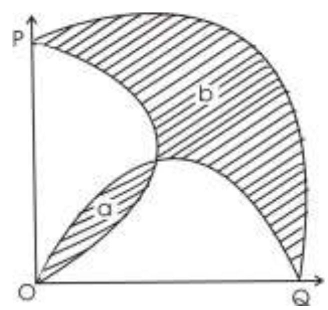
(i) $\sqrt{x}+\frac{1}{\sqrt{x}}$ (ii) $x^3+\frac{1}{x^3}$

4. (a) Solve :

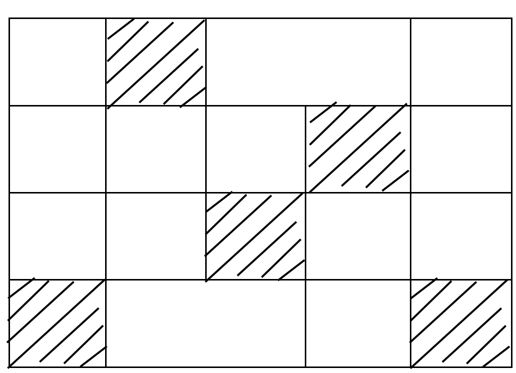
$$\frac{x^2-1}{x^2-4}-\frac{x^2-5}{x^2-8}=\frac{x^2-2}{x^2-5}-\frac{x^2-6}{x^2-9}$$

(b) Find the remainder when $x^{81}+x^{49}+x^{25}+x^9+x$ is divided by x^3-x.

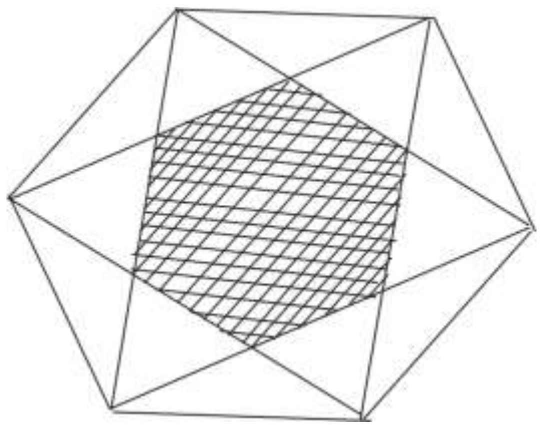
5. (a) OPQ is a quadrant of a circle and semicircles are drawn on OP and OQ. Areas a and b are shaded. Find a/b.



(b) Assuming all vertical lines are parallel, all angles are right angles and all the horizontal lines are equally spaced, what fraction of figure is shaded ?



6. Alternate vertices of a regular hexagon are joined as shown. What fraction of the total area of a hexagon is shaded ? (Justify your answer)

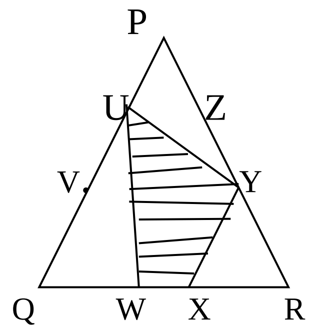


7. In a competition consisting of 30 problems Neeta was given 12 points for each correct solution, and 7 points were subtracted from her score for each incorrect solution problems not attempted contributed 0 points to the score find the number of problems attempted correctly by Neeta.

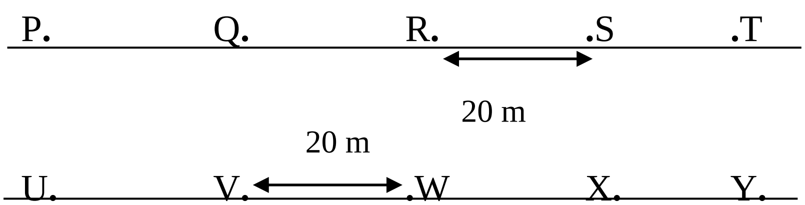
8. A cube with each edge of lengths 4 units is painted green on all the faces. The cube is then cut into 64 unit cubes. How many of these small cubes have (i) 3 faces painted (ii) 2 faces painted (iii) one face painted (iv) no face painted.

9. Let PQR be an equilateral triangle with each side of length 3 units. Let U, V, W, X Y and Z divide the sides into unit lengths. Find the ratio of the area U, W, X, Y and Z divide the sides into unit lengths. Find the ratio of

the area U W X Y (shaded) to the area of the whole triangle PQR.



10. Five houses P, Q, R, S and T are situated on the opposite side of a street from five other houses U, V, W, X and Y as shown in the diagram :



Houses on the same side of the street are 20 metres apart A postman is trying to decide whether to deliver the letters using route PQRSTYXWVU or route PUQVRWSXTY, and finds that the total distance is the same in each case. Find the total distance in metres.

KV JMO 2003 **SOLUTIONS**

Q.1.(a) The digits in ascending order are = 2789

The digits is decending order are = 9872

$\therefore$ their difference = 9872 – 2789

= 7083

(b) Speed = 12 km/h

$\therefore$ Speed in metres per minutes $\frac{\overset{200}{\cancel{12} \times \cancel{200}}}{1 \times \underset{5}{\cancel{60}}}$

$\therefore$ He travels 200 m per minute.

(c) The square root of 35 x 65 x 91

= $\sqrt{35 \times 65 \times 91}$

= $\sqrt{5 \times 7 \times 5 \times 13 \times 7 \times 13}$

= $\sqrt{5^2 \times 7^2 \times 13^2}$

= 5 x 7 x 13 = 455

(d) Let, the number = x

$\therefore$ 15% of x = 81

$\Rightarrow$ $\frac{15}{100} x = 81$

$\Rightarrow$ $x = \frac{\overset{27}{\cancel{81}} \times \overset{20}{\cancel{20}}}{\underset{5}{\cancel{15}}}$

= 540

∴ The req. number = 540 .

(e) Difference between 9:45 A.M. and 12:58 P.M.

12 : 58 PM

-9 : 45 AM

3h 13 min

∴ Difference = 3h 13 min

= 3 x 60 + 13

= 180 + 13

= 193 min.

Q2.(a) 203203

= 7 x 7 x 11 x 13 x 29

= 7^2 x 11 x 13 x 29

∴ largest prime factor of 203203 = 29

(b) To find the last two digits of $(2003)^{2003}$

$2003 \equiv 3 \pmod{100}$

⇒ $2003^2 \equiv 9 \pmod{100}$

⇒ $2003^4 \equiv 81 \pmod{100}$

$\equiv -19 \pmod{100}$ (i)

$\Rightarrow 2003^8 \equiv (-19)(-19) \pmod{100}$

$\equiv 361 \pmod{100}$

$\equiv -39 \pmod{100}$

$\Rightarrow 2003^{16} \equiv (-39)(-39) \pmod{100}$

$\equiv 1321 \pmod{100}$

$\equiv 21 \pmod{100}$ (ii)

Multiplying (i) and (ii),

$2003^{20} \equiv (-19)(21) \pmod{100}$

$\equiv -299 \pmod{100}$

$\equiv 1 \pmod{100}$

$\therefore 2003^{20} \equiv 1 \pmod{100}$

$\Rightarrow 2003^{2000} \equiv 1 \pmod{100}$

$\Rightarrow (2003)^{2000} \times (2003)^3 \equiv 1 \times 3 \times 3 \times 3 \pmod{100}$

$\Rightarrow (2003)^{2003} \equiv 27 \pmod{100}$

$\therefore$ The tens and units digits of $(2003)^{2003}$ are 2 and 7.

Q.3 (a) The numbers are :

$3^3 \times 1^3 = 27 > 1$

$3^3 \times 2^3 = 216 > 1$

$3^3 \times 3^3$

$3^3 \times 4^3$

: x :

: x :

$3^3 \times 31^3$

$3^3 \times 32^3$

$3^3 \times 33^3 = 970309 < 1000009$

$3^3 \times 34^3 = 1061208 > 1000009$

$\therefore$ There are 33 numbers satisfying the given condition.

3. (b)

(ii) $x^3 + \frac{1}{x^3} = ?$

$$x + \frac{1}{x} = 5 + 2\sqrt{6} + \frac{1}{5 + 2\sqrt{6}} \times \frac{5 - 2\sqrt{6}}{5 - 2\sqrt{6}}$$

$$= 5 + 2\sqrt{6} + \frac{5 - 2\sqrt{6}}{5^2 - (2\sqrt{6})^2}$$

$$= 5 + 2\sqrt{6} + \frac{5 - 2\sqrt{6}}{25 - 24}$$

$$= 5 + 2\sqrt{6} + 5 - 2\sqrt{6}$$

$$= 10$$

$$\therefore\ x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3.(x)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right)$$

$= 10^3 – 3 \text{ x } 10$

$= 1000 – 30$

$= 970$

$$\therefore\ x^3 + \frac{1}{x^3} = 970$$

Q3. (b) $x = 5 + 2\sqrt{6}$

(i) $\sqrt{x} + \dfrac{1}{\sqrt{x}} = ?$

$$\Rightarrow \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = x + \frac{1}{x} + 2.\sqrt{x}\ x\left(\frac{1}{\sqrt{x}}\right)$$

$$= \quad 5 + 2\sqrt{6} + \frac{1}{5 + 2\sqrt{6}} + 2$$

$$= \quad 5 + 2\sqrt{6} + \frac{1}{5 + 2\sqrt{6}}\ x\ \frac{5 - 2\sqrt{6}}{5 - 2\sqrt{6}} + 2$$

$$= 7 + 2\sqrt{6} + \frac{5 - 2\sqrt{6}}{5^2 - (2\sqrt{6})^2}$$

$$= 7 + 2\sqrt{6} + \frac{5 - 2\sqrt{6}}{25 - 24}$$

$$= 7 + \cancel{2\sqrt{6}} + 5 - \cancel{2\sqrt{6}}$$

$$= 12$$

$$\therefore \sqrt{x} + \frac{1}{\sqrt{x}} = \sqrt{12} = 2\sqrt{3}$$

Q4. (a) $\dfrac{x^2 - 1}{x^2 - 4} - \dfrac{x^2 - 5}{x^2 - 8} = \dfrac{x^2 - 2}{x^2 - 5} - \dfrac{x^2 - 6}{x^2 - 9}$

$$\Rightarrow \quad \frac{(x^2-1)(x^2-8)-(x^2-5)(x^2-4)}{(x^2-4)(x^2-8)}$$

$$= \quad \frac{(x^2-2)(x^2-9)-(x^2-6)(x^2-5)}{(x^2-5)(x^2-9)}$$

$$\Rightarrow \quad \frac{x^4-9x^2+8-[x^4-9x^2+20]}{x^4-12x^2+32}$$

$$\Rightarrow \frac{x^4-11x^2+18-[x^4-11x^2+30]}{x^4-14x^2+45}$$

$$\Rightarrow \frac{-12}{x^4-12x^2+32} = \frac{-12}{x^4-14x^2+45}$$

$$\Rightarrow x^4-14x^2+45 = x^4-12x^2+32$$

$$\Rightarrow -14x^2+12x^2 = 32 - 45$$

$$\Rightarrow \not{-}2x^2 = \not{-}13$$

$$\Rightarrow x = \pm\sqrt{\frac{13}{2}}$$

$$\therefore x = +\sqrt{\frac{13}{2}} \text{ or } -\sqrt{\frac{13}{2}}$$

4. (b) $x^{81} + x^{49} + x^{25} + x^9 + x$

$= x\,(x^{80} + x^{48} + x^{24} + x^8 + 1)$

And $x^3 - x$

$= x(x^2-1)$

Dividing = $x^{80} + x^{48} + x^{24} + x^8 + 1$ by x^2-1, we get,

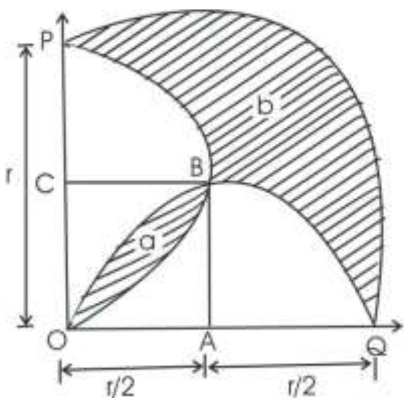
Quotient = $X^{78} + x^{76} + \ldots\ldots + x^{48} + 2x^{46} + 2x^{44} + \ldots.. + 2x^{24} + 3x^{22} + 3x^{20} +$

$\ldots\ldots + 3x^8 + 4x^6 + 4x^4 + 4x^2 + 4$

Divisor = $x^2 - 1$

Then, remainder = 5

Q5. (a)



ar (OABC) = $\left(\frac{r}{2}\right)^2$

$= \frac{r^2}{4}$

$\therefore$ ar (Sector ABO) = $\frac{1}{4}\pi\left(\frac{r}{2}\right)^2$

$= \frac{1}{4}\pi\frac{r^2}{4}$

$= \frac{\pi r^2}{16}$

$\therefore$ ar (Sector BOC) = ar (OABC – Sector ABO)

$$= \frac{r^2}{4} - \frac{\pi r^2}{16}$$

$$= \frac{4r^2 - \pi r^2}{16}$$

$$\therefore \text{ ar (region a)} = \frac{r^2}{4} - 2\left(\frac{4r^2 - \pi r^2}{16}\right)$$

$$= \frac{r^2}{4} - \frac{4r^2 - \pi r^2}{8}$$

$$= \frac{2r^2 - 4r^2 - \pi r^2}{8}$$

$$= \frac{\pi r^2 - 2r^2}{8}$$

$$\therefore \text{ ar (region b)} = \frac{\pi r^2}{4} - \left(\frac{r^2}{4} + \frac{2.\pi r^2}{16}\right)$$

$$= \frac{\pi r^2}{4} - \frac{r^2}{4} + \frac{\pi r^2}{8}$$

$$= \frac{2\pi r^2 - 2r^2 + \pi r^2}{8}$$

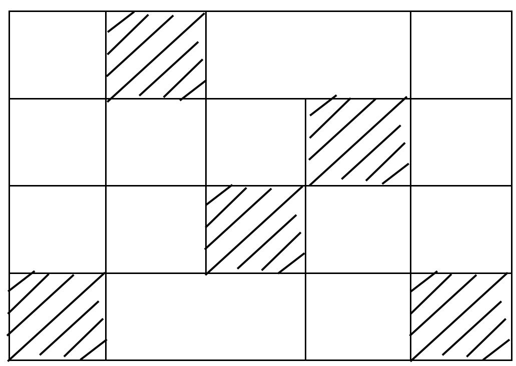
$$= \frac{\pi r^2 - 2r^2}{8}$$

$$\therefore \quad \frac{a}{b} = \frac{\frac{\pi r^2 - 2r^2}{8}}{\frac{\pi r^2 - 2r^2}{8}}$$

= 1

$$\therefore \frac{a}{b} = 1$$

5. (b)



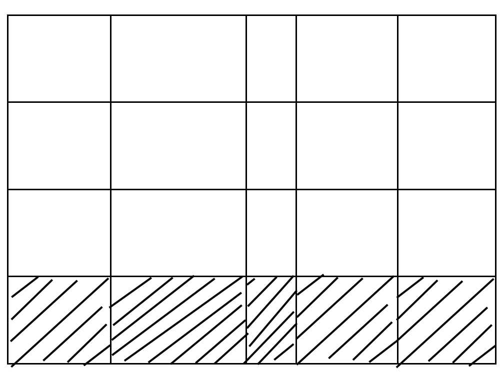
Bringing down all the shaded portions in one line, we have,

¼

¼

¼

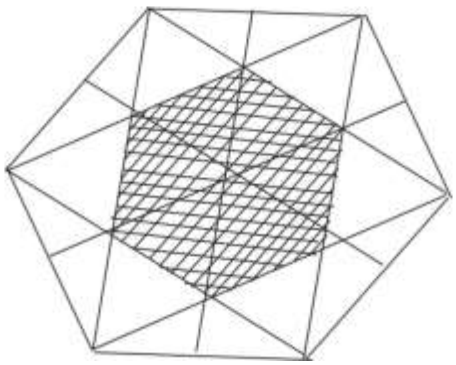
¼

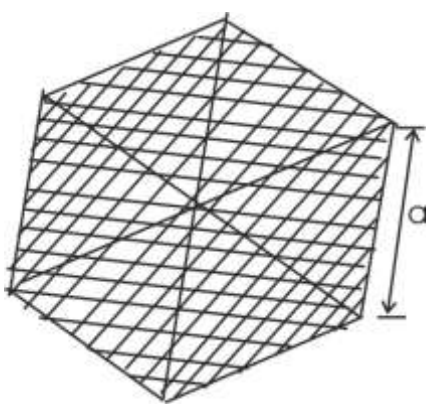


The shaded portion covers exactly ¼ th of the total area.

∴ ¼ th portion of the figure is shaded.

Q6.





So, it is clear from the above figure that, the regular hexagon is divided into 12 equilateral congruent triangles and 12 other triangles of which when two are taken they form one triangle of area equal to one triangle of inner hexagon.

∵ The shaded region covers 6 triangles.

∴ The whole hexagon covers $\left(12+\frac{\overset{6}{12}}{2}\right)$ triangles i.e. 18 triangles.

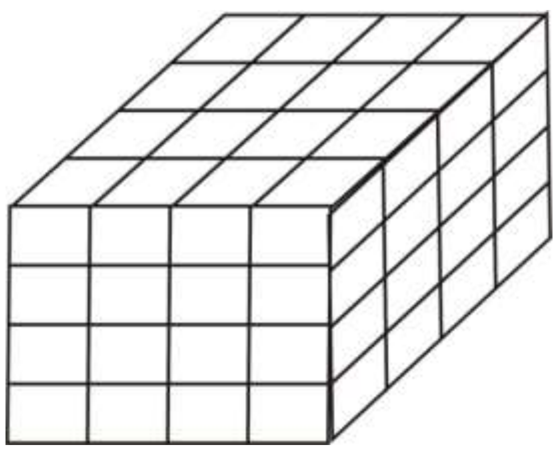
∴ Fraction of the shaded area 6/18th of the hexagon

= 1/3rd of the bigger hexagon

Q7. INSUFFICIENT INFORMATION GIVEN

SOLUTION NOT POSSIBLE

Q8.



(i) Cubes with 3 faces painted.

No. of cubes with 3 faces painted = 8

(ii) Cubes with 2 faces painted.

No. of cubes with 2 faces painted = 24

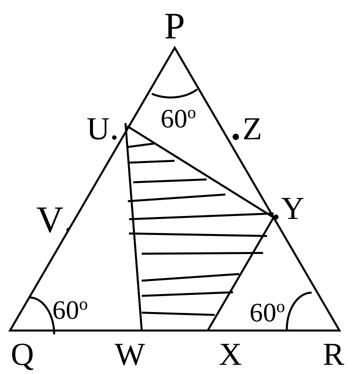
(iii) Cubes with 1 face painted.

No. of cubes with 1 face painted = 24

(iv) Cubes with no face painted.

No. of cubes with no faces painted = 8

Q9.



In ΔUQW,

$UQ = 2$

$QW = 1$

$\angle UQW = 60^o$

$\frac{QW}{UQ} = \frac{1}{2} = \cos 60$

$= \cos \angle UQW$

Hence, ΔUWQ must be right triangle, with $\angle QWU = 90^o$

$\therefore UW = \sqrt{UQ^2 - QW^2}$

$= \sqrt{2^2 - 1^2}$

$= \sqrt{4 - 1}$

$= \sqrt{3}$

$\therefore$ $(\Delta UWQ) = \frac{1}{2}$ x QW x UW

$= \frac{1}{2} x 1 x \sqrt{3}$

$= \frac{\sqrt{3}}{2}$ sq.units

Similarly,

In ΔPUY,

PU = 1

PY = 2

$\angle$ UPY = 60^o

$\therefore$ $\frac{PU}{PY} = \frac{1}{2} = \cos 60$

$= \cos \angle$ UPY

$\therefore$ ΔPUY must be right triangle, with $\angle$ PUY=90^o

$\therefore$ UY = $\sqrt{PY^2 - PU^2}$

$= \sqrt{2^2 - 1^2}$

$= \sqrt{4 - 1}$

$= \sqrt{3}$

$\therefore$ ar $(\Delta PUY) = \frac{1}{2}$ x PU x UY

$= \frac{1}{2} \text{x} 1 \text{x} \sqrt{3}$

$= \frac{\sqrt{3}}{2} \text{sq.units}$

Also,

ar (XYR) = $\frac{\sqrt{3}}{2} \text{x} 1^2$

$= \frac{\sqrt{3}}{4} \text{sq.units}$

ar (ΔABC) = $\frac{\sqrt{3}}{4} \text{x} 3^2$

$= \frac{9\sqrt{3}}{4} \text{sq.units}$

$\therefore$ ar (UWXY) = ar (ΔABC)- $\left[\text{ar}(PUY) + \text{ar}(UWQ) + \text{ar}(XYR) \right]$

$= \frac{9\sqrt{3}}{4} - \left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} \right]$

$= \frac{9\sqrt{3}}{4} - \left[\frac{2\sqrt{3} + 2\sqrt{3} + \sqrt{3}}{4} \right]$

$= \frac{9\sqrt{3}}{4} - \frac{5\sqrt{3}}{4}$

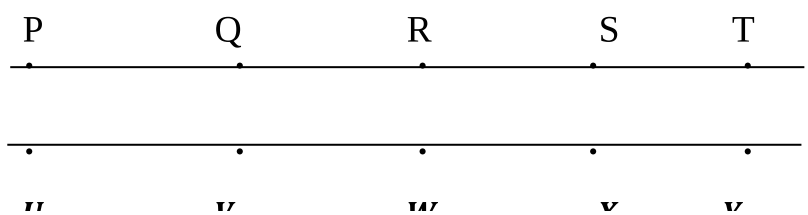
$= \frac{9\sqrt{3} - 5\sqrt{3}}{4}$

$= \frac{4\sqrt{3}}{4}$

$= \sqrt{3}$ sq.units

$\therefore$ ar (UWXY) = $\sqrt{3}$ sq.units

Q10.



Let, PV = QW = RX = SY = TX

= SW = RV = QU = x

and PU = TY = y

$\therefore$ In case of route P Q R S T Y X W V U,

Total Distance = PQ + QR + RS + ST + TY + YX + XW + WV + VU

= 20 + 20 + 20 + 20 + y + 20 + 20 + 20 + 20

= 160 + y

In case of route P U Q V R W S X T Y,

Total Distance = PU + UQ + QV + VR + RW + WS + SX + XT + TY

= y + x + y + x + y + x + y + x + y

= 4x + 5y

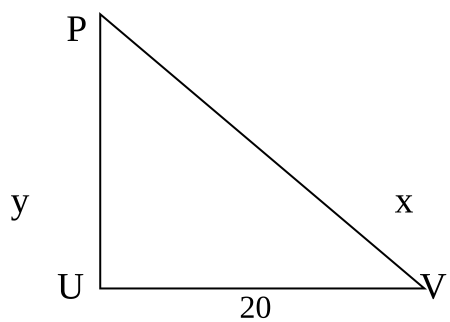
given,

160 + y = 4x + 5y

$\Rightarrow$ 160 = 4x + 4y

$\Rightarrow \quad x + y = 40 \qquad\qquad\qquad\qquad$ (i)

Considering ΔPUV



By Pythagoras theorem,

$x^2 - y^2 = 20^2$

$\Rightarrow \quad (x+y)\,(x-y) = 400$

$$\Rightarrow \quad x + y = \frac{400}{x - y}$$

$$\Rightarrow \quad \cancel{40} = \frac{\overset{10}{\cancel{400}}}{x - y} \qquad\qquad [\because x + y = 10 \text{ from (i)}]$$

$\Rightarrow x - y = 10 \qquad\qquad\qquad\qquad$ (ii)

$x + y = 40$

$x - y = 10$

$\underline{\;-\quad +\quad -\;\;}$

$2y = 30$

$\Rightarrow y = 15$

Total distance = 160 + y

= 160 + 15

= 175 m