KVS Junior Mathematics Olympiad (JMO) – 2001

M.M. 100 Time : 3 hours

Note : (i) Please check that there are two printed pages and ten question in all.

(ii) Attempt all questions. All questions carry equal marks.

1. Fill in the blanks :

   (a) If $x + y = 1$, $x^3 + y^3 = 4$, then $x^2 + y^2 =$ ……..

   (b) After 15 litres of petrol was added to the fuel tank of a car, the tank was 75% full. If the capacity of the tank is 28 litres, then the number of litres in the tank before adding the petrol was ……

   (c) The perimeter of a rectangle is 56 metres. The ratio of its length to width is 4:3. The length of the diagonal in metres is ……..

   (d) If April 23 falls on Tuesday, then March 23 of the same year was a ……..

   (e) The sum of the digits of the number $2^{2000}5^{2004}$ is ….

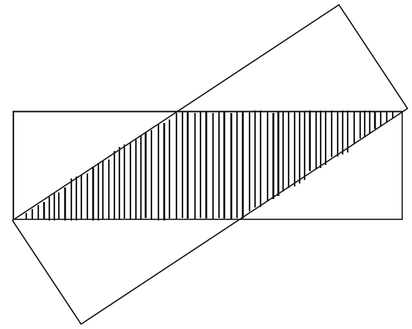
2. (a) Arrange the following in ascending order :

   2^{5555}, 3^{3333}, 6^{2222}

   (b) Two rectangles, each measuring 3 cm x 7 cm,

   are placed as in the adjoining figure :

Find the area of the overlapping portion (shaded) in cm^2.



3. (a) Solve :

$$\frac{\log_{10}(35-x^3)}{\log_{10}(5-x)} = 3$$

(b) Simplify :

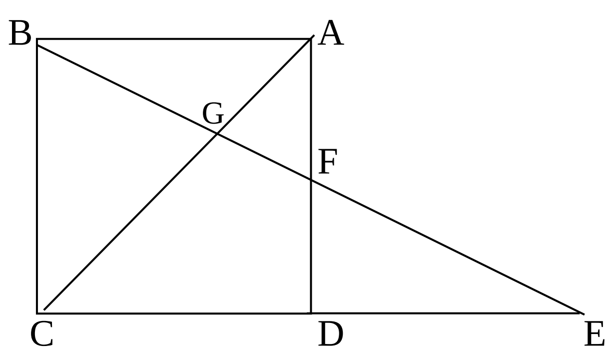
$$\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$$

4. (a) Factorize :

$(x-y)^3+(y-z)^3+(z-x)^3$

(b) If $x^2-x-1=0$, then find the value of x^3-2x+1

5. ABCD is a square. A line through B intersects CD produced at E, the side AD at F and the diagonal AC at G.



If BG = 3, and GF=1, then find the length of FE,

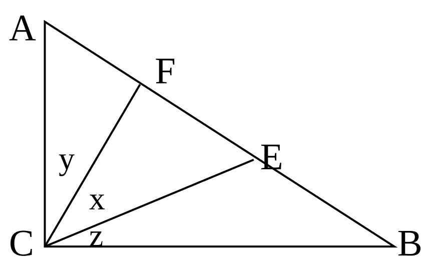
6. (a) Find all integers n such that $(n^2-n-1)^{n+2} = 1$

(b) If $x = \frac{4ab}{a+b}$, find the value of $\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b}$

7. (a) Find all the positive perfect cubes that divide 9^9.

(b) Find the integer closest to $100\,(12-\sqrt{143})$

8. In a triangle ABC, $\angle BCA=90^o$. Points E and F lie on the hypotenuse AB such that AE=AC and BF = BC. Find $\angle ECF$.



9. An ant crawls 1 centimetre north, 2 centimetres west, 3 centimetres south, 4 centimetres east, 5 centimetres north and so on, at 1 centimetre per second. Each segment is 1 centimetre longer than the preceding one, and at the end of a segment, the ant makes a left turn. In which direction is the ant moving 1 minute after the start ?

10. Find the lengths of the sides of a triangle with 20, 28 and 35 as the lengths of its altitudes.

SOLUTION KV JMO 2001

Q1.

(i) 7

(ii) 6 litres

(iii) 20 m

(iv) Friday

(v) 13

Q2.

(a) $2^{5555}, 3^{3333}, 6^{2222}$

$2^5 = 32$

$3^3 = 27$

$6^2 = 36$

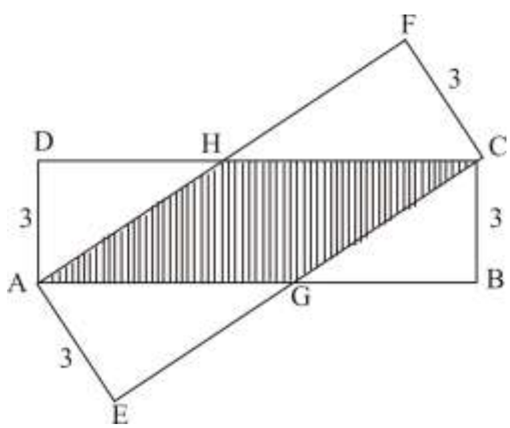
$\therefore 3^3 < 2^5 < 6^2$

$\therefore \quad 3^{3333} < 2^{5555} < 6^{2222}$

$\therefore$ The required order of the three numbers is :-

$3^{3333}, 2^{5555}, 6^{2222}$

Q2(b)



∵ Two rectangles each measuring 3cm x 7 cm are placed in this manner.

∴The four triangles formed must be congruent to each other.

∴ $\Delta EAG \cong \Delta BCG$

∴AG = BG

and EG = CG

Let, BG = x

∴AG = AB – BG

= 7 – x

∴CG = AG = 7-x

∴ In ΔBCG, we have,

BC = 3

BG = x

CG = 7 – x

∴By Pythagoras theorem, In ΔBCG, we have,

$CG^2 = BC^2 + BG^2$

$\Rightarrow (7-x)^2 = 3^2 + x^2$

$\Rightarrow 49 - 14x + x^2 = 9 + x^2$

$\Rightarrow 49 - 9 = 14x$

$\Rightarrow 40 = 14x$

$\Rightarrow x = \frac{40}{14}$

$\Rightarrow x = \frac{20}{7}$

$\therefore AG = 7 - x$

$= 7 - \frac{20}{7}$

$= \frac{49 - 20}{7}$

$= \frac{29}{7}$

$\because$ DC || AB

$\Rightarrow$ HC || AG

Similarly AH || GC

$\therefore$ AG CH is a parallelogram

$\therefore$ Area of the AGCH = (Base) x (Height)

= AG x BC

$= \frac{29}{7} \times 3$

$= \frac{87}{7}$

$= 12\frac{3}{7}\text{cm}^2$

3.(a) $\frac{\log_{10}(35 - x^3)}{\log_{10}(5 - x)} = 3$

$\Rightarrow \log_{10}(35 - x^3) = 3.\log_{10}(5-x)$

$\Rightarrow \log_{10}(35 - x^3) = \log_{10}(5-x)^3$

$\Rightarrow 35 - x^3 = (5-x)^3$

$\Rightarrow 35 - \cancel{x^3} = 125 - \cancel{x^3} + 3(5)(-x)(5-x)$

$\Rightarrow 35 = 125 - 15x(5-x)$

$\Rightarrow 35 = 125 - 75x + 15x^2$

$\Rightarrow 15x^2 - 75x + 125 - 35 = 0$

$\Rightarrow 15x^2 - 75x + 90 = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 3x - 2x + 6 = 0$

$\Rightarrow x(x - 3) - 2(x - 3) = 0$

$\Rightarrow (x - 3)(x - 2) = 0$

$\therefore x = 3 \text{ or } 2$

Q3. (b)

$$\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$$

= (a-b) (b+c) (c+a) + (b-c) (a+b) (c+a)

$$= \frac{+(c-a)(a+b)(b+c)+(a-b)+(b-c)(c-a)}{(a+b)(b+c)(c+a)}$$

= (c+a) [(a-b) (b+c) +(b-c) (a+b)]

$$= \frac{+(c-a)[(a-b)+(b-c)+(a+b)(b+c)]}{(a+b)(b+c)(c+a)}$$

= (c+a) [ab + ac - b^2 – bc + ab + b^2 – ac - bc]

$$= \frac{+(c-a)[ab+ac-b^2+bc+ab+b^2-ac+bc]}{(a+b)(b+c)(c+a)}$$

$$= \frac{(c+a)(2ab-2bc)+(c-a)(2ab+2bc)}{(a+b)(b+c)(c+a)}$$

$$= \frac{2abc-2bc^2+2a^2b-2bc+2abc+2bc^2-2a^2b-2abc}{(a+b)(b+c)(c+a)}$$

$$= \frac{0}{(a+b)(b+c)(c+a)}$$

= 0

4. (a)

= $(x-y)^3 + (y-z)^3 + (z-x)^3$

$= (x – y + y - z) [(x-y)^2 + (y-z)^2 + (x-y) (y-z)] + (z-x)^3$

$= (x –z) [x^2+y^2-2xy+y^2+z^2-2yz+xy-xz-y^2+yz] + (z-x)^3$

$= (x –z) (x^2+y^2+z^2 –xy-yz-zx) - (x-z)^3$

$= (x –z) [x^2+y^2+z^2 –xy-yz-zx) - (x-z)^2]$

$= (x –z) [\cancel{x^2}+y^2+\cancel{z^2} –xy-yz-zx – \cancel{x^2}-\cancel{z^2} +2xz]$

$= (x-z) [y^2-yz-xy+xz]$

$= (x-z) [y(y-z)-x(y-z)]$

$= (x-z)(y-x)(y-z)$

$=(x-y) (y-z) (z-x)$

4. (b) $x^2 – x – 1 = 0$

$x^3 – 2x + 1 = ?$

Dividing $(x^3 – 2x + 1)$ by $(x^2 – x - 1)$, we get,

$$
\begin{array}{r|l}
 & \quad x+1 \\
x^2-x-1 & x^3-2x+1 \\
 & \underset{-}{x^3} \underset{+}{-} x^2 \underset{+}{-} x \\
\hline
 & x^2-x+1 \\
 & \underset{-}{x^2} \underset{+}{-} x \underset{+}{+} 1 \\
\hline
 & \quad 2
\end{array}
$$

$\therefore x^3 – 2x + 1 = (x^2-x-1) (x+1) + 2$

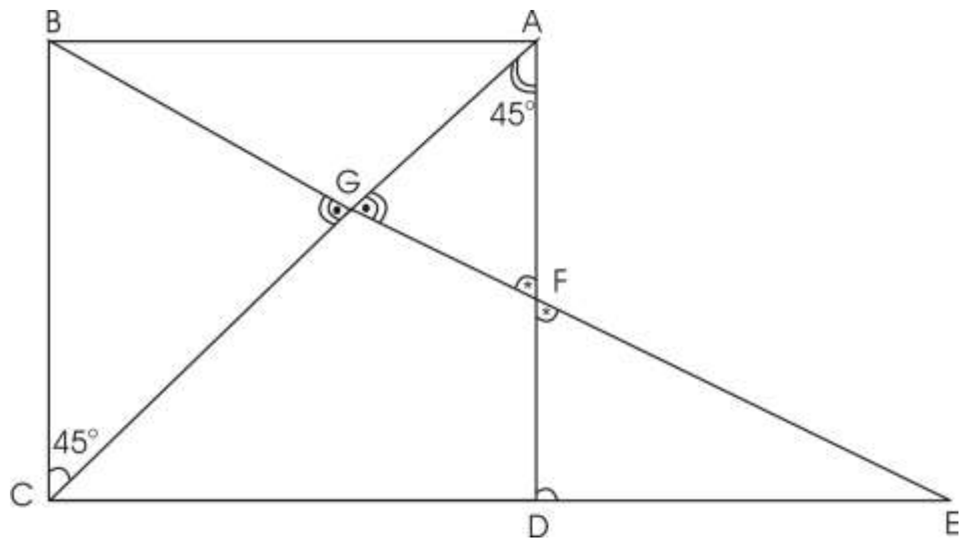
$\Rightarrow x^3 - 2x + 1 = 0 \times (x + 1) + 2$

$\Rightarrow x^3 - 2x + 1 = 0 + 2$

$\Rightarrow x^3 - 2x + 1 = 2$

$\therefore\ x^3 - 2x + 1 = 2$

Q5



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Given :

ABCD is a square

BG = 3

GF = 1

BE = ?

Let, AB = BC=CD = DA = a

Now,

In ΔBCG and ΔFAG,

$\angle BCG = \angle FAG$ (Each = 45°)

$\angle BGC = \angle AGF$ (Vertically opposite angles)

$\therefore$ ΔBCG ~ ΔFAG by AA rule

$$\therefore \frac{BC}{AF} = \frac{BG}{GF}$$

$$\Rightarrow \frac{a}{AF} = \frac{3}{1}$$

$$\Rightarrow AF = \frac{a}{3}$$

$\therefore$ DF = AD – AF = $a - \frac{a}{3} = \frac{2}{3}a$

Now,

In ΔABF and ΔDEF,

$\angle A = \angle D = 90^o$

$\angle AFB = \angle DFE$ (Vertically opposite angles)

$\therefore$ ΔABF ~ ΔDEF by AA rule

$$\therefore \frac{BF}{AF} = \frac{EF}{DF}$$

$$\Rightarrow \frac{BF}{\frac{a}{3}} = \frac{EF}{\frac{2}{3}a}$$

$$\Rightarrow 2.BF = EF$$

$\Rightarrow$ 2 X 4 = EF

$\Rightarrow$ EF = 8

$\therefore$ BE = BF + FE

= 4+ 8

= 12 units.

Q6.(a) $(n^2-n-1)^{n+2} = 1$

$\Rightarrow (n^2-n-1)^{n+2} = (1)^{n+2}$

$\Rightarrow n^2 - n - 1 = 1$

$\Rightarrow n^2 - n - 2 = 0$

$\Rightarrow n^2 - 2n + n - 2 = 0$

$\Rightarrow n(n - 2) + 1(n - 2) = 0$

$\Rightarrow (n - 2)(n + 1) = 0$

$\therefore$ n = 2 or -1

Q6. (b) $x = \frac{4ab}{a+b}$

$$\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b} = ?$$

$$\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b}$$

$$= \frac{x+2a}{x-2a} + 1 + \frac{x+2b}{x-2b} - 1$$

$$= \frac{x+2a+x-2a}{x-2a} + \frac{x+2b-x+2b}{x-2b}$$

$$= \frac{2x}{x-2a} + \frac{4b}{x-2b}$$

$$= \frac{2x}{x-2a} - 2 + \frac{4b}{x-2b} + 2$$

$$= \frac{2x-2x+4a}{x-2a} + \frac{4b+2x-4b}{x-2b}$$

$$= \frac{4a}{x-2a} + \frac{2x}{x-2b}$$

$$= \frac{4a}{\frac{4ab}{a+b}} + \frac{2x\frac{4ab}{a+b}}{\frac{4ab}{a+b} - 2b}$$

$$= \frac{4a(a+b)}{4ab-2a^2-2ab} + \frac{8ab}{4ab-2ab-2b^2}$$

$$= \frac{4a^2+4ab}{2ab-2a^2} + \frac{8ab}{2ab-2b^2}$$

$$= \frac{2a+2b}{b-a} + \frac{4a}{a-b}$$

$= \frac{-2a - 2b + 4a}{a - b}$

$= \frac{2a - 2b}{a - b}$

$= \frac{2(a - b)}{(a - b)}$

= 2

Q7. (a) 9^9

$= (3^3)^9$

$= (3)^{27}$

The positive perfect cubes that divide 9^9 are :

$1^3, 3^3, (3^2)^3, (3^3)^3, (3^4)^3, (3^5)^3, (3^6)^3, (3^7)^3, (3^9)^3$.

i.e. 10 numbers

(b) 100 $(12-\sqrt{143})$

$\because \sqrt{143} < \sqrt{144}$

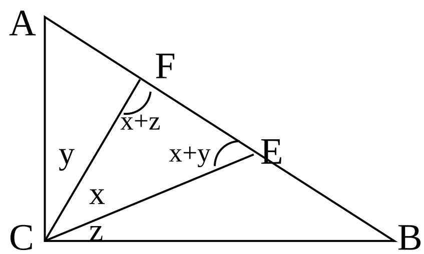
i.e. $\sqrt{143} < 12$

and $\sqrt{143}$ is less than 12 by a very small margin.

$\therefore$ The closest integer to 100 $(12 - \sqrt{143})$

is , 100.

8.



Given :

$\angle BCA = 90^o$

AE = AC

BF = BC

$\angle ECF = ?$

$\because \angle BCA = 90$

$\Rightarrow x + y + z = 90^o$

In ΔACE

AE = AC

$\therefore \angle AEC = \angle ACE = x + y$

In ΔBCF,

$\because$ BF = CF

$\therefore \angle BCF = \angle BFC = x + z$

$\therefore$ In ΔCFE,

$$\angle FCE + \angle CFE + \angle CEF = 180$$

$\Rightarrow x + x + z + x + y = 180$

$\Rightarrow$ 2x + x + y + z = 180

$\Rightarrow$ 2x + 90 = 180

$\Rightarrow$ 2x = 90

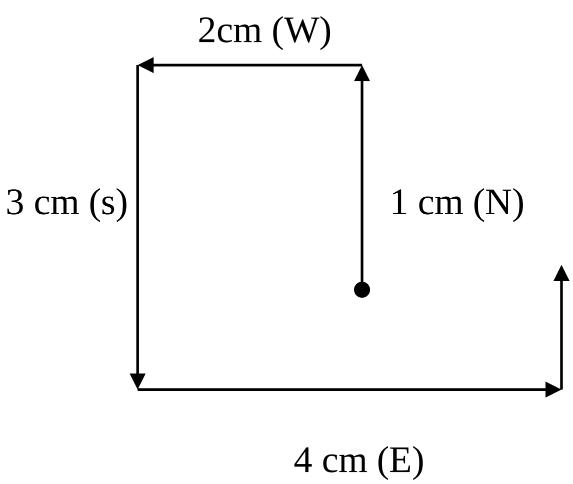
$\Rightarrow$ x = 45^o

$\therefore \angle$ECF = x

= 45^o

$\therefore \angle$ECF = 45^o

9.



So, we get that,

The ant always travels (4k+1) cm North,

The ant always travels (4k+2) cm West,

The ant always travels (4k+3) cm South,

The ant always travels (4k+4) cm East,

$\therefore$ In 1 min the ant travels, distance = 60 x 1 cm

= 60 cm

We get the following AP,

1, 2, 3, …….

a = 1

d = 1

$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow 60 = \frac{n}{2}[2 \times 1 + (n-1) \times 1]$

$\Rightarrow 120 = n[2 + n-1]$

$\Rightarrow 120 = n[n+1]$

$\Rightarrow 120 = n^2 + n$

$\Rightarrow n^2 + n - 120 = 0$

$$\therefore n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-120)}}{2x1}$$

$$= \frac{-1 \pm \sqrt{1 + 480}}{2}$$

$$\frac{-1 \pm \sqrt{481}}{2}$$

$$\frac{-1 \pm 22}{2} \quad [22^2 = 484]$$

$\frac{21}{2}$ [Taking the positive value]

$= 10\frac{1}{2}$

But we'll have to take, n = 11

$\therefore$ a_{11} = 1 + 10 x 1

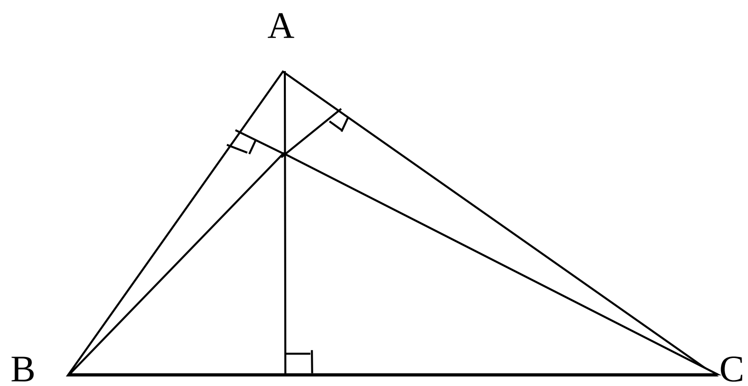
= 1 + 10

= 11

= 4 x 2 + 3

$\therefore$ The ant was traveling towards South.

Q10.



$h_a = 20$

$h_b = 28$

$h_c = 35$

$\Delta = \frac{1}{2} a \text{ x } h_a \Rightarrow a = \frac{2\Delta}{h_a}$

$= \frac{2\Delta}{20} = \frac{\Delta}{10}$

$\Delta = \frac{1}{2} b \text{ x } h_b \Rightarrow b = \frac{2\Delta}{h_b}$

$= \frac{2\Delta}{28} = \frac{\Delta}{14}$

$\Delta = \frac{1}{2} c \text{ x } h_c \Rightarrow c = \frac{2\Delta}{h_c} = \frac{2\Delta}{35}$

$\therefore\ a = \frac{2\Delta}{20}$

$b = \frac{2\Delta}{28}$

$c = \frac{2\Delta}{35}$

$\therefore\ S = \frac{a+b+c}{2} = \frac{1}{2}\left[\frac{2\Delta}{20} + \frac{2\Delta}{28} + \frac{2\Delta}{35}\right]$

$= \Delta\left[\frac{1}{20} + \frac{1}{28} + \frac{1}{35}\right]$

$S\text{-}a = \Delta\left[\frac{1}{20} + \frac{1}{28} + \frac{1}{35}\right] - \frac{2\Delta}{20}$

$$= \Delta \left[\frac{1}{28} + \frac{1}{35} - \frac{1}{20} \right]$$

$$= \Delta \left[\frac{35x20 + 28x20 - 28x35}{20x28x35} \right]$$

$$= \Delta \left[\frac{700 + 560 - 980}{20x28x35} \right]$$

$$= \Delta \left[\frac{\overset{10}{\cancel{280}}}{\underset{2}{\cancel{20}}x\cancel{28}x35} \right]$$

$$= \Delta \left[\frac{1}{70} \right]$$

$$= \frac{\Delta}{70}$$

S-b $= \Delta \left[\frac{1}{20} + \frac{1}{28} + \frac{1}{35} \right] - \frac{2\Delta}{28}$

$$= \Delta \left[\frac{1}{20} + \frac{1}{35} - \frac{1}{28} \right]$$

$$= \Delta \left[\frac{35x28 + 28x20 - 20x35}{20x28x35} \right]$$

$$= \Delta \left[\frac{980 + 560 - 700}{20x28x35} \right]$$

$$= \Delta \left[\frac{\overset{42}{\cancel{840}}\,\cancel{0}^{3}}{\cancel{20} \text{ x } \underset{14}{\cancel{28}} \text{ x } \underset{5}{\cancel{35}}} \right]$$

$$= \Delta \left[\frac{3}{70} \right]$$

$$= \frac{3\Delta}{70}$$

$$\text{S-c} \quad = \Delta \left[\frac{1}{20} + \frac{1}{28} + \frac{1}{35} \right] - \frac{2\Delta}{35}$$

$$= \Delta \left[\frac{1}{20} + \frac{1}{28} - \frac{1}{35} \right]$$

$$= \Delta \left[\frac{35\text{x}28 + 20\text{x}35 - 20\text{x}28}{20\text{x}28\text{x}35} \right]$$

$$= \Delta \left[\frac{980 + 700 - 560}{20\text{x}28\text{x}35} \right]$$

$$= \Delta \left[\frac{\overset{56}{\cancel{1120}}^{2}}{\cancel{20}\text{x}\cancel{28}\text{x}35} \right]$$

$$= \Delta \left[\frac{2}{35} \right]$$

$$= \frac{2\Delta}{35}$$

$$\therefore \Delta \quad = \quad \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{\Delta\left[\frac{1}{20}+\frac{1}{28}+\frac{1}{35}\right]\left[\frac{\Delta}{70}\right]\left[\frac{3\Delta}{70}\right]\left[\frac{2\Delta}{35}\right]}$$

$$= \Delta^2\sqrt{\frac{2240}{20x28x35}x\frac{1}{70}x\frac{3}{70}x\frac{2}{35}}$$

$$= \Delta^2\sqrt{\frac{16x3}{28x35x10x35x35}}$$

$$= \Delta^2\sqrt{\frac{6}{35x35x35x35}}$$

$$= \Delta^2\frac{\sqrt{6}}{35x35}$$

$$\Rightarrow \Delta = \Delta^2\ \frac{\sqrt{6}}{35x35} =$$

$$\therefore\ \Delta = \frac{35x35}{\sqrt{6}}$$

$$\therefore\ a = \frac{2\Delta}{20}$$

$$a = \frac{245\sqrt{6}}{6}$$

$$b = \frac{2\Delta}{28}$$

$$b = \frac{175\sqrt{6}}{12}$$

$c = \frac{2\Delta}{35} = \frac{35\sqrt{6}}{3}$ units.