

Indian Olympiad Qualifier in Astronomy (IOQA) 2020 – 2021

conducted jointly by
Homi Bhabha Centre for Science Education (HBCSE-TIFR)
and
Indian Association of Physics Teachers (IAPT)

Part II: Indian National Astronomy Olympiad (INAO) Homi Bhabha Centre for Science Education (HBCSE-TIFR)

Question Paper

Date: 6th February 2021

Roll Number: - -

Time: 10:15 – 12:15 hrs

Maximum Marks: 80

Please Note:

- Please write your roll number in the space provided above.
- There are total 5 questions. Maximum marks are indicated in front of each sub-question.
- For all questions, the process involved in arriving at the solution is more important than the final answer. Valid assumptions / approximations are perfectly acceptable. Please write your method clearly, explicitly stating all reasoning / assumptions / approximations.
- Use of non-programmable scientific calculators is allowed.
- **The answer-sheet must be returned to the invigilator.** You can take this question paper back with you.

Useful Constants

Mass of the Sun	$M_{\odot} \approx 1.989 \times 10^{30} \text{ kg}$
Mass of the Earth	$M_{\oplus} \approx 5.972 \times 10^{24} \text{ kg}$
Mass of the Moon	$M_{\zeta} \approx 7.347 \times 10^{22} \text{ kg}$
Radius of the Earth	$R_{\oplus} \approx 6.371 \times 10^6 \text{ m}$
Speed of Light	$c \approx 2.998 \times 10^8 \text{ m s}^{-1}$
Radius of the Sun	$R_{\odot} \approx 6.955 \times 10^8 \text{ m}$
Radius of the Moon	$R_m \approx 1.737 \times 10^6 \text{ m}$
Distance of the Moon from Earth	$d_{\zeta} \approx 3.844 \times 10^8 \text{ m}$
Astronomical Unit	$a_{\oplus} \approx 1.496 \times 10^{11} \text{ m}$
Gravitational Constant	$G \approx 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$

1. A Newtonian reflector type of telescope has a concave mirror with a 2.00 m radius of curvature as its primary mirror. It is fitted with camera at primary focus with an achromatic camera lens of 4.00 cm focal length.
 - (a) (2 marks) What is the angular magnification of this system?
 - (b) (3 marks) We observe a 25 000 km diameter sunspot with this system. What will be the angular size of the sunspot in the image?
 - (c) (7 marks) The camera lens is now removed and an image detector is placed in such a way that a well-focused sunspot image due to primary mirror could be observed with yellow light (wavelength 550 nm). We wish to observe the same region now in green light (wavelength 465 nm). For this, we introduce a green filter, which blocks all other wavelengths except green light, in front of the detector. If the thickness of this plane parallel glass plate (refractive index 1.53) is $t = 2.887$ mm, how much will be the change in the position of the image?
2. A rectangular sheet of paper was rolled to form a cylinder, with exactly two layers of sheet along the curved surface. This cylinder is cut such that the plane of the cut makes an angle of 45° with the axis of cylinder. The paper was then unrolled and spread on a flat table.
 - (a) (2 marks) Draw a figure to show how the unrolled paper would appear.
 - (b) (4 marks) Justify your answer with appropriate mathematical arguments.
3. Betelgeuse, a red supergiant star, in the constellation of Orion, is known as an irregular variable star. It's magnitude varies between +0.3 to +1.0 from time to time. However, last year the astronomers were surprised to observe an unexpected dimming of Betelgeuse. We may assume this event started from 12 October 2019. Given below is a plot of the observed magnitude vs time (light-curve) of Betelgeuse.

<https://gofacademy.in>

Notes:

- The relation between magnitude of the star and light flux received from it is given by :

$$m_1 - m_2 = -2.5 \log_{10} \left(\frac{F_1}{F_2} \right)$$

where m_1 and m_2 are magnitudes measured in two different observations and F_1 and F_2 are corresponding light fluxes.

- Mass of Betelgeuse: $M_B = 2.1 \times 10^{31}$ kg
 - Distance of Betelgeuse from the Earth: $d_B = 200$ pc
 - Typical radius of Betelgeuse: $R_1 = 6.17 \times 10^{11}$ m
- (a) (8 marks) One of the proposed model for this dimming was that the whole star suddenly started expanding and hence cooled down. Let us assume that the star is still acting as perfect black body at each stage of expansion (and subsequent contraction). By other measurements, we know that the star's effective temperature at the start of expansion was $T_1 = 3500$ K and the effective temperature at the most expanded state is $T_2 = 2625$ K. Find the average velocity of the expansion of gas.
 - (b) (9 marks) Some other astronomers proposed that the said dimming is caused due to the transit of a giant exoplanet with radius ' r ' orbiting Betelgeuse. Argue if such a scenario is possible for an edge-on circular orbit of the exoplanet with orbital radius ' a '.

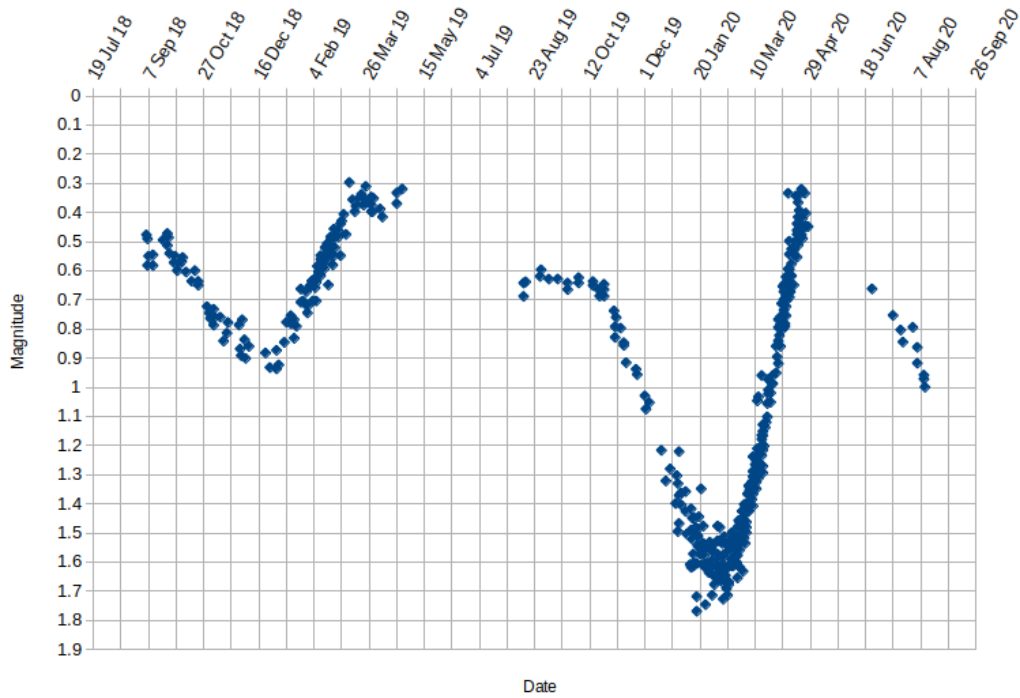


Figure 1: The V band magnitudes are observations from the AAVSO International Database (<https://www.aavso.org>)

- (c) (7 marks) A popular model to explain this dimming states that this event started with a large plume of hot material getting ejected from the star's surface. This material cooled down after ejection and became opaque to block light from a part of the star. As this dense cloud expanded, it kept blocking more and more part of the star dimming it further. However, as this expansion lowered the density of the cloud, the cloud's opaqueness started reducing after a few weeks and the star started brightening again.

Here we will consider a simpler version of this model. We assume that this material ejection happened in a narrow cone in very short timescale from a single point on the non-rotating stellar surface. The total mass of the ejected material was approximately equal to mass of the earth and the axis of the cone was exactly along our line of sight. Let us assume that at each instant during the expansion the density of material is constant throughout the cone and the vertex of the cone is still attached to the stellar surface.

We assume that the star starts brightening again when the average density inside the cone falls to $5 \times 10^{-14} \text{ kg m}^{-3}$. Find the time averaged velocity of particles, which form the front of the expanding cone.

4. (20 marks) A space agency would like to put an artificial satellite in a highly elliptical orbit around the Earth in the ecliptic plane (the plane of the Earth's orbit around the Sun). What can be the maximum eccentricity (e_{max}) for such an orbit? Also provide the perigee (r_{min}) and apogee (r_{max}) distances (in km) from the centre of the Earth.

5. (18 marks) Five friends from various cities of India observed the Sun and made the following statements. For their observations of shadows they all used a metre stick placed vertically on a flat ground.
1. I observed sunrise at 04:56 on 12th June.
 2. I observed sunrise at 05:24 on 12th June, which was the second earliest out of the five cities on that day.
 3. I observed sunset at 16:55 on 24th December.
 4. I observed sunset at 17:35 on 24th December, which was the third earliest out of the five cities on that day.
 5. I observed sunset at 18:50 on 1st September, which was the last sunset on that day.
 6. At local noon (i.e. noon as per each local time) on 21st June, shadow at my location was the longest amongst all.
 7. The shortest shadow of the year at my location was observed on 21st June.
 8. The shortest shadow of the year at my location was observed on 5th June.
 9. The shortest shadow of the year at my location was observed on 26th May.
 10. The shortest shadow of the year at my location was observed on 15th April.
 11. On 1st July, I had a longer day as compared to other observers.
 12. On 1st February, I had a longer day as compared to other observers.

Below are the locations of our observers along with the coordinates of their cities;

Observer	Location	Coordinates
Kamal	Kolkata	22.57° N, 88.36° E
Naeem	Nagpur	21.15° N, 79.09° E
Chandrika	Chandigarh	30.73° N, 76.78° E
Kate	Kochi	9.93° N, 76.27° E
Mayank	Mumbai	19.08° N, 72.88° E

Assume that all observers have their watches synchronised to correct Indian Standard Time. Find out, for each statement, which statement was made by which observer?

Note: You don't have to give reasons. Only a table with observer name and statement numbers is enough. Each correct pair gives you 1.5 marks. However, for each wrong pair, you will lose 0.5 mark.

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Part II: Indian National Physics Olympiad (INPhO)

Homi Bhabha Centre for Science Education (HBCSE-TIFR)

Date: 07 February 2021

Time: 10:15-12:15 (2 hours)

Maximum Marks: 50

Instructions

1. This booklet consists of 10 pages and total of 5 questions. Write roll number at the top wherever asked.
2. Booklet to write the answers is provided separately. Instructions to write the answers are on the Answer Booklet.
3. Marks will be awarded on the basis of what you write on both the Summary Answer Sheet and the Detailed Answer Sheets in the Answer Booklet. Simple short answers and plots may be directly entered in the Summary Answer Sheet. Marks may be deducted for absence of detailed work in questions involving longer calculations.
4. Strike out any rough work that you do not want to be considered for evaluation. You may also use the space on the Question Paper for rough work – this will NOT be evaluated.
5. Non-programmable scientific calculators are allowed. Mobile phones **cannot** be used as calculators.
6. Last page of the question paper can be used for rough work.
7. **Please submit the Answer Booklet at the end of the examination.** You may retain the Question Paper.

Table of Constants

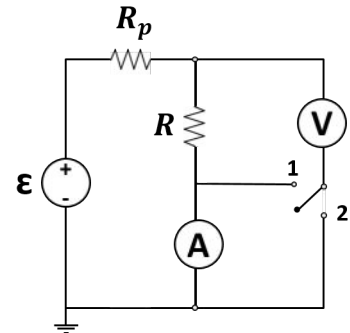
Speed of light in vacuum	c	$3.00 \times 10^8 \text{ m}\cdot\text{s}^{-1}$
Planck's constant	h	$6.63 \times 10^{-34} \text{ J}\cdot\text{s}$
Magnitude of electron charge	e	$1.60 \times 10^{-19} \text{ C}$
Rest mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg}$
Value of $1/4\pi\epsilon_0$		$9.00 \times 10^9 \text{ N}\cdot\text{m}^2\cdot\text{C}^{-2}$
Acceleration due to gravity	g	$9.81 \text{ m}\cdot\text{s}^{-2}$

Please note that alternate/equivalent methods and different way of expressing final solutions may exist. A correct method will be suitably awarded.

1. The ammeter-voltmeter method is widely used for measuring electrical resistances in the physics laboratory. In this method, the resistance R is always derived from the readings V and I from a voltmeter and an ammeter respectively, using Ohm's law: $R = V/I$. While using this method, it is assumed that the ammeter and voltmeter used in the setup are ideal. In this problem, we will find the pitfalls of this assumption and devise a new setup with a better performance.

The standard ammeter-voltmeter setup consists of a DC voltage source (ε) maintained at a constant voltage, a protection resistance (R_p), an ammeter (A), and a voltmeter (V). The unknown internal resistances of the ammeter and the voltmeter are R_A and R_V , respectively. Also, $R_V \gg R_A$. We aim to measure the true value R of an unknown resistor.

We consider a two commonly used circuit configurations (1) and (2) indicated by the two possible positions of the switch in the circuit diagram shown below. Let the measured values of the resistance R be R_{m1} and R_{m2} in the setups (1) and (2), respectively. The relative error, Δ , is defined as the ratio of the absolute error of the measurement to the actual value: $\Delta = (R_m - R)/R$.



- (a) [2 marks] Obtain the relative errors in the measurements (Δ_1 and Δ_2) for each of the above configurations.

Solution:

Setup (1):

Current through the ammeter

$$I_1 = \frac{V_1}{R} + \frac{V_1}{R_V}$$

whereas the voltage corresponds to the voltage across the voltmeter-resistance combination. Thus the measured resistance

$$R_{m1} = \frac{V_1}{I_1} = \frac{R}{1 + \frac{R}{R_V}} \quad (1.1)$$

$$\Delta_1 = -\frac{1}{1 + \frac{R_V}{R}} \quad (1.2)$$

Setup (2):

Current through the ammeter

$$I_2 = \frac{V_2}{R + R_A}$$

Thus

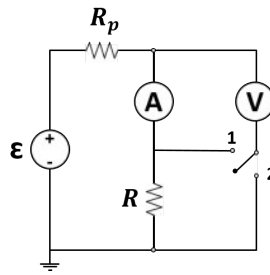
$$R_{m2} = \frac{V_2}{I_2} = R + R_A \quad (1.3)$$

$$\Delta_2 = \frac{R_A}{R} \quad (1.4)$$

- (b) [4 marks] Using exactly the same circuit elements, can you suggest a step by step procedure, with the necessary circuit diagram(s), to measure the true value of the resistance R , regardless of the values of the internal resistances of the ammeter and the voltmeter? You may use the measurements made in part (a).

Solution:

1. We use the configuration shown below. When the switch is in position 1, ammeter and voltmeter are in parallel and we calculate $R_A = V/I$.



2. When we use the switch in position 2, the configuration is same as position 2 of part (a).

$$R = R_{m2} - R_A$$

This will give the true value of the resistance.

2. [8 marks] Prof. Saha gave the following problem to four students.

In this problem work done by a system on its surroundings is taken as positive. A non-ideal gas follows the Van der Waals equation of state

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P , V , and T denote the pressure, volume, and temperature, respectively; n is the number of moles; R is the universal gas constant and a , b are dimensional positive constants. This gas expands adiabatically from an initial temperature T_i and volume V_i to a final temperature T_f and volume V_f . The adiabatic process is described by an equation of the form $f(P, V; n, a, b, \alpha) = \text{constant}$, where α is a dimensionless number which is greater than 1. It is given that $\alpha \rightarrow \gamma$ in the ideal gas limit, where γ is the adiabatic exponent. What is the work (W) done by the gas in the process?

The four students solved the problem independently and gave four different answers. Their answers were:

$$\begin{aligned} \text{(a)} \quad W &= \frac{nR}{\alpha - 1}(T_i - T_f) + n^2 a(V_f^{-1} - V_i^{-1}) & \text{(c)} \quad W &= \frac{nR}{\alpha - 1}(T_i - T_f) + n^2 a(V_f^{\alpha-1} - V_i^{\alpha-1}) \\ \text{(b)} \quad W &= \frac{nR}{\alpha - 1}(T_f - T_i) + n^2 a(V_f^{-1} - V_i^{-1}) & \text{(d)} \quad W &= \frac{nR}{\alpha - 1}(T_i - T_f) \left[1 - \left(\frac{V_f - nb}{V_i - nb}\right)^{\alpha-1} \right] \end{aligned}$$

Now, Prof. Saha had actually provided the exact expression of $f(P, V; n, a, b, \alpha)$ to the students, but could not remember it during evaluation. Still, he could determine that some or all of the four answers above must be incorrect, based on general physical arguments alone.

Consider each of the four answers and give at least one reason for each of them showing why it is wrong, or possibly correct. *Note that you are not required to give a correct expression for W or a detailed derivation for it in this question.*

Solution:

- (a) For adiabatic expansion, $T_f < T_i$, and $V_f > V_i$. Even though the first term is positive and the second term is negative, it is possible to have $W > 0$, which is true for adiabatic expansion. Also, in the ideal gas limit ($\alpha \rightarrow \gamma$ and $a \rightarrow 0$), this gives the correct expression. So this may be the correct expression.
- (b) For adiabatic expansion, $T_f < T_i$ and $V_f > V_i$. Therefore, in this case $W < 0$, which is incorrect.

- (c) From Van der Waals equation, a/V has dimensions of energy. So $aV^{\alpha-1}$ cannot have dimensions of energy, making this expression incorrect.
- (d) Since $T_f < T_i$ and $V_f > V_i$, here $W < 0$ making this incorrect.

3. Consider an electron (mass m , magnitude of charge e) moving initially around a nucleus of charge $2e$ in a circular orbit of radius 10^{-10} m. In this problem we use SI units throughout and neglect all relativistic effects.

- (a) [2 marks] Obtain the expression for the frequency, f , of the electron in the circular orbit (numerical value is not required).

Solution:

The centripetal force for the circular motion of the electron is provided by the Coulomb attraction of the nucleus. Let r be the radius of the circular orbit, and v the speed of the electron in this orbit, then

$$\frac{mv^2}{r} = \frac{2e^2}{4\pi\epsilon_0 r^2} \quad (3.1)$$

$$f = \frac{v}{2\pi r} = \left(\frac{2}{4\pi\epsilon_0 m} \right)^{1/2} \frac{e}{2\pi r^{3/2}} \quad (3.2)$$

From classical electrodynamics, we know that an accelerated electron radiates energy. The expression for the power P of this radiation is given by

$$P = K\epsilon_0^w e^x a^y c^z$$

where a is the acceleration, c is the speed of light, ϵ_0 is the permittivity of free space, and K is a dimensionless constant.

- (b) [2 marks] Obtain $\{w, x, y, z\}$ using dimensional analysis.

Solution: $w = -1, x = 2, y = 2, z = -3$

Due to the loss of energy through radiation, the electron does not remain in the circular orbit, and gradually spirals into the nucleus. Take the constant K to be 5.31×10^{-2} .

- (c) [5 marks] Let T be the time it takes for the electron to reach the nucleus. Calculate T if the radius of the nucleus is 10^{-14} m.

Solution:

The total energy of an electron in the orbit is

$$E(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r} \quad (3.3)$$

$$-\dot{E}(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2 \dot{r}}{r^2} \quad (3.4)$$

the acceleration is

$$a = \frac{v^2}{r} = \frac{1}{4\pi\epsilon_0} \frac{2e^2}{mr^2} \quad (3.5)$$

We use Eq. (3.5) in the power radiated, which yields the energy loss rate

$$-\dot{E}(r) = -K \frac{1}{(4\pi\epsilon_0)^2} \frac{4e^6}{\epsilon_0 c^3 m^2 r^4} \quad (3.6)$$

Here negative sign indicates that the energy of the electron is decreasing. Combining

Eqs. (3.4) and (3.6)

$$r^2 dr = -\frac{K4e^4}{(4\pi\epsilon_0)\epsilon_0 c^3 m^2} dt \quad (3.7)$$

Integrating the equation

$$\int_{10^{-10}}^{10^{-14}} r^2 dr = -\frac{K4e^4}{(4\pi\epsilon_0)\epsilon_0 c^3 m^2} \int_0^T dt \quad (3.8)$$

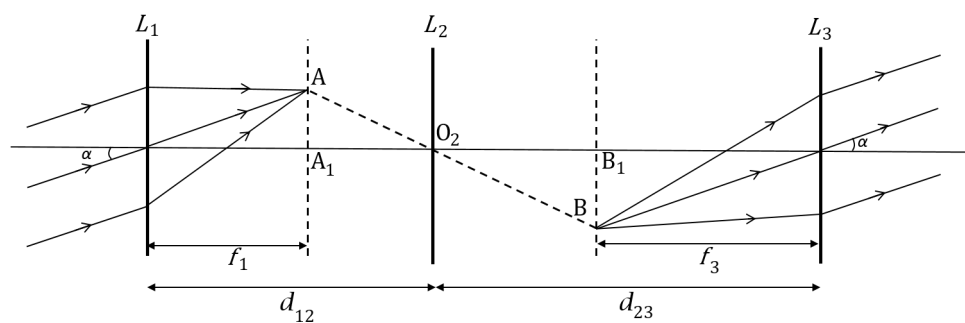
which yields

$$T \sim \frac{10^{-30}}{48\pi} \frac{(4\pi\epsilon_0)^2 c^3 m^2}{K e^4} \sim 5.26 \times 10^{-11} \text{ s.} \quad (3.9)$$

4. [12 marks] Three thin convex lenses L_1 , L_2 , and L_3 with focal lengths f_1 , f_2 , and f_3 , respectively, are arranged in order (L_1 followed by L_2 , followed by L_3 from left to right) with their principal axes coincident. The distance d_{12} between L_1 and L_2 , and the distance d_{23} between L_2 and L_3 are such that $d_{12} + d_{23} \geq f_1 + 4f_2 + f_3$. If a parallel beam of light incident on L_1 at a small angle to the principal axis remains parallel to itself when leaving the system after passing through L_2 and L_3 , draw the appropriate ray diagram and determine d_{12} and d_{23} in terms of f_1 , f_2 , and f_3 .

Solution:

An incoming parallel beam falling on the thin lens L_1 will converge to a certain point A on the focal plane of L_1 . The point A serves as the point source for L_2 whose image is formed on the other side of L_2 at a certain point B. The line AB must intersect the principal axis at the pole O_2 of L_2 . For a parallel beam to emerge from L_3 , B must lie in the focal plane of L_3 . The necessary ray diagram is drawn below.



Since $\alpha \approx 0$, we make the approximations

$$AA_1 = f_1 \tan \alpha \approx f_1 \alpha$$

$$BB_1 = f_2 \tan \alpha \approx f_2 \alpha$$

From magnification formula for lens L_2 ,

$$\frac{BB_1}{AA_1} = \frac{f_3 \alpha}{f_1 \alpha} = \frac{v_2}{-u_2} = \frac{d_{23} - f_3}{d_{12} - f_1}$$

$$\implies \frac{d_{12}}{f_1} = \frac{d_{23}}{f_3} = k(\text{say})$$

$$\implies d_{12} = k f_1 \quad \text{and} \quad d_{23} = k f_3$$

From the lens equation for lens L_2 ,

$$\frac{1}{v_2} - \frac{1}{u_2} = \frac{1}{f_2}$$

$$\frac{1}{(d_{23} - f_3)} - \frac{1}{-(d_{12} - f_1)} = \frac{1}{f_2}$$

$$\frac{1}{f_3(k-1)} + \frac{1}{f_1(k-1)} = \frac{1}{f_2}$$

$$\Rightarrow k = 1 + \frac{f_2}{f_1} + \frac{f_2}{f_3}$$

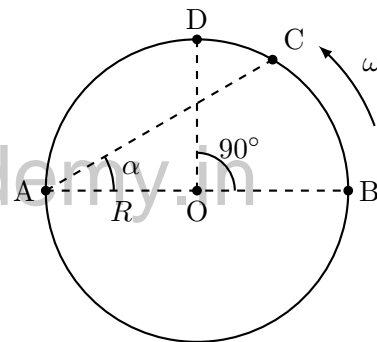
Then,

$$d_{12} = f_1 + f_2 + \frac{f_1 f_2}{f_3}$$

$$d_{23} = f_2 + f_3 + \frac{f_2 f_3}{f_1}$$

5. Two friends, Amina (A) and Beena (B), are sitting at diametrically opposite points of a merry-go-round (taken as a circular disk in the horizontal plane) of radius R that is rotating at constant angular speed ω in the anticlockwise direction, when viewed from the top (see figure below).

When Amina is at the position A (as shown in the figure), she throws a ball with velocity \vec{u} (relative to the merry-go-round) in such a manner that Beena catches it when she reaches the position C ($\angle BAC = \alpha$). Here \vec{u} makes an angle θ with respect to the horizontal, and ϕ is the angle made by the horizontal projection of \vec{u} with respect to the line AB. Neglect air resistance, friction, and the effect of throwing or catching the ball on the speed of the merry-go-round.



Top view

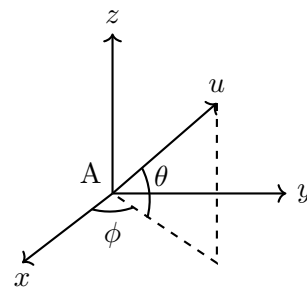
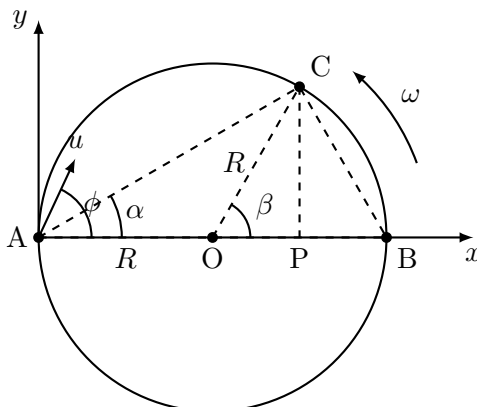
- (a) [6 marks] Determine u , θ and ϕ , in terms of R , ω , α , and other relevant quantities.

Solution:

Point of throwing: A; Point of catching: C

Position of C at instant of projection: B

We take the point A as the origin and the x -axis along the diameter AB. The y -axis is in the horizontal plane, perpendicular to AB. The z -axis is taken along vertical direction.



Given, ω = angular speed of rotation; \vec{u} = velocity of throwing

θ = Projection angle with respect to horizontal

ϕ = Projection angle with respect to diameter AB (x -axis)

$\alpha = \angle BAC \implies \beta = \angle BOC = 2\alpha$

Time of flight = time taken for B to reach $C = T = \frac{R\beta}{v_s} = \frac{R\beta}{R\omega} = \frac{2\alpha}{\omega}$

Equations of motion along three directions:

$$\begin{aligned} x: \quad u_x \cdot T &= AP \\ \implies (u \cos \theta \cos \phi) \cdot \frac{2\alpha}{\omega} &= R + R \cos \beta = R(1 + \cos 2\alpha) = 2R \cos^2 \alpha \\ \implies u \cos \theta \cos \phi &= \frac{R\omega}{\alpha} \cos^2 \alpha \end{aligned} \quad (5.1)$$

$$\begin{aligned} y: \quad (u_y - R\omega) \cdot T &= CP \\ \implies (u \cos \theta \sin \phi - R\omega) \cdot \frac{2\alpha}{\omega} &= R \sin \beta = 2R \sin \alpha \cos \alpha \\ \implies u \cos \theta \sin \phi &= \frac{R\omega}{\alpha} \sin \alpha \cos \alpha + R\omega = \frac{R\omega}{\alpha} [\sin \alpha \cos \alpha + \alpha] \end{aligned} \quad (5.2)$$

$$\begin{aligned} z: \quad u_z \cdot T - \frac{1}{2}gT^2 &= 0 \\ \implies u \sin \theta &= \frac{gT}{2} = \frac{g\alpha}{\omega} \end{aligned} \quad (5.3)$$

Dividing eq. (5.2) by eq. (5.1),

$$\begin{aligned} \tan \phi &= \frac{\sin \alpha \cos \alpha + \alpha}{\cos^2 \alpha} = \tan \alpha + \alpha \sec^2 \alpha \\ \implies \phi &= \tan^{-1}(\tan \alpha + \alpha \sec^2 \alpha) \end{aligned} \quad (5.4)$$

Squaring eqs. (5.1), (5.2), (5.3) and adding,

$$\begin{aligned} u^2 \cos^2 \theta + u^2 \sin^2 \theta &= \left(\frac{R\omega}{\alpha}\right)^2 [\cos^4 \alpha + \sin^2 \alpha \cos^2 \alpha + \alpha^2 + 2\alpha \sin \alpha \cos \alpha] + \left(\frac{g\alpha}{\omega}\right)^2 \\ \implies u^2 &= \left(\frac{R\omega}{\alpha}\right)^2 [\cos^2 \alpha + 2\alpha \sin \alpha \cos \alpha + \alpha^2] + \left(\frac{g\alpha}{\omega}\right)^2 \end{aligned} \quad (5.5)$$

$$\implies u = \left[\left(\frac{g\alpha}{\omega}\right)^2 + \left(\frac{R\omega}{\alpha}\right)^2 [\cos^2 \alpha + 2\alpha \sin \alpha \cos \alpha + \alpha^2] \right]^{1/2} \quad (5.6)$$

From (5.3) and (5.6),

$$\theta = \sin^{-1} \left[\frac{g\alpha}{\omega} \left[\left(\frac{g\alpha}{\omega}\right)^2 + \left(\frac{R\omega}{\alpha}\right)^2 [\cos^2 \alpha + 2\alpha \sin \alpha \cos \alpha + \alpha^2] \right]^{-1/2} \right] \quad (5.7)$$

- (b) [3 marks] If Amina throws the ball with $\phi = 60^\circ$, and appropriate values of θ and u such that Beena can catch it, what is the magnitude of the displacement, s , of the ball when it is caught by Beena? For this part only, take $R = 1.5$ m, and it is enough to state your answer within a range of 0.5 m.

Solution:

The displacement of the ball is the length of $AC = s = 2R \cos \alpha$.

Thus we need to determine α when $\phi = 60^\circ$. Equation (5.4) can be used for this. Note that values of θ and u are not needed.

Putting $\phi = 60^\circ$ in equation (5.4), we have

$$f(\alpha) = \tan \alpha + \alpha \sec^2 \alpha = \tan 60^\circ = \sqrt{3}$$

This equation cannot be solved analytically. We use trial values of α to find the solution

by interpolation.

$$f(\pi/6) = \frac{1}{\sqrt{3}} + \frac{\pi}{6} \left(\frac{2}{\sqrt{3}} \right)^2 = 1.275 < \sqrt{3}$$

$$f(\pi/4) = 1 + \frac{\pi}{4} (\sqrt{2})^2 = 2.571 > \sqrt{3}$$

Thus

$$\frac{\pi}{6} < \alpha < \frac{\pi}{4}$$

$$\Rightarrow \frac{\sqrt{3}}{2} > \cos \alpha > \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2R \frac{\sqrt{3}}{2} > 2R \cos \alpha > 2R \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sqrt{3}R > s > \sqrt{2}R$$

Putting $R = 1.5$ m,

$$\boxed{2.1 \text{ metre} < s < 2.6 \text{ metre}}$$

Any answer that encloses the actual value of 2.4 m and has a range ≤ 0.5 m is acceptable.

- (c) [0.5 marks] Determine the speed of throwing u_D if Beena catches the ball at the point D ($\angle BOD = 90^\circ$), instead of C.

Solution:

This is a special case of the above, where $\alpha = \frac{\pi}{4}$. Using the above results,

$$u_D = \left[\left(\frac{g\pi}{4\omega} \right)^2 + \left(\frac{4R\omega}{\pi} \right)^2 \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right] \right]^{1/2} \quad (5.8)$$

- (d) [3 marks] What should be the angular speed ω_m of the merry-go-round for which the speed of throwing u_D will be minimum for Beena to catch the ball at the position D? What is this minimum speed of throwing u_m ?

Solution:

This can be determined by finding the minimum of u_D , or equivalently, u_D^2 . From (5.8),

$$u_D^2 = \left(\frac{g\pi}{4\omega} \right)^2 + \left(\frac{4R\omega}{\pi} \right)^2 \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right]$$

$$\therefore \frac{d(u_D^2)}{d\omega} \Big|_{\omega_m} = 0 \Rightarrow -\frac{(g\pi)^2}{8\omega_m^3} + \frac{32R^2\omega_m}{\pi^2} \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right] = 0$$

$$\Rightarrow \omega_m^4 = \frac{g^2\pi^4}{256R^2 \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right]}$$

$$\Rightarrow \omega_m = \frac{\pi}{4} \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right]^{-1/4} \sqrt{\frac{g}{R}}$$

Also,

$$\frac{d^2(u_D^2)}{d\omega^2} \Big|_{\omega_m} = \frac{3(g\pi)^2}{8\omega_m^4} + \frac{32R^2}{\pi^2} \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right] > 0.$$

implying u_D^2 is minimum at $\omega = \omega_m$.

$$\therefore u_m^2 = gR \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right]^{1/2} + gR \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right]^{1/2}$$

$$\Rightarrow u_m = \left(\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right)^{1/4} \sqrt{2gR}$$

Alternative solution without calculus

Observe that

$$u_D^2 = \frac{\lambda}{\omega^2} + \mu\omega^2$$

where $\lambda = \left(\frac{g\pi}{4} \right)^2 > 0$, $\mu = \left(\frac{4R}{\pi} \right)^2 \left[\frac{1}{2} + \frac{\pi}{4} + \frac{\pi^2}{16} \right] > 0$.

We can write

$$u_D^2 = \left(\frac{\sqrt{\lambda}}{\omega} - \omega\sqrt{\mu} \right)^2 + 2\sqrt{\lambda\mu}.$$

The first term can be made zero by the choice of

$$\omega = \omega_m = \left(\frac{\lambda}{\mu} \right)^{1/4}$$

leading to the minimum value of u_D^2 as $u_m^2 = 2\sqrt{\lambda\mu}$. Upon substituting the values of λ and μ the desired expressions are obtained.

- (e) [2.5 marks] Consider the case when Amina throws the ball when she is at A, and catches it herself when she reaches the point B (Beena is not involved in this case). Take the angular speed of the merry-go-round to be $\omega = \sqrt{g/R}$. Find u , θ and ϕ in this case.

Solution:

This case is NOT a special case of the above.

Now $T = \frac{\pi}{2} = \frac{\pi}{\omega}$.

Further, $\omega = \sqrt{g/R}$.

The equations of motion are:

$$x: u_x \cdot T = AB \Rightarrow (u \cos \theta \cos \phi) \cdot \frac{\pi}{\omega} = 2R \Rightarrow u \cos \theta \cos \phi = \frac{2R\omega}{\pi} = \frac{2}{\pi} \sqrt{gR} \quad (5.9)$$

$$y: (u_y - R\omega) \cdot T = 0 \Rightarrow u \cos \theta \sin \phi = R\omega = \sqrt{gR} \quad (5.10)$$

$$z: u_z T - \frac{1}{2}gT^2 = 0 \Rightarrow u \sin \theta = \frac{gT}{2} = \frac{\pi g}{2\omega} = \frac{\pi}{2} \sqrt{gR} \quad (5.11)$$

Dividing eq. (5.10) by eq. (5.9),

$$\tan \phi = \frac{\pi}{2} \Rightarrow \phi = \tan^{-1} \frac{\pi}{2} = 57.52^\circ$$

Squaring eqs. (5.10) and (5.9), and adding,

$$u^2 \cos^2 \theta = gR \left[\frac{4}{\pi^2} + 1 \right]$$

$$\Rightarrow u = \sqrt{gR} \left[\frac{\pi^2}{4} + \frac{4}{\pi^2} + 1 \right]^{1/2} = 1.97 \sqrt{gR} \quad (5.12)$$

Using eqs. (5.11) and (5.12),

$$\sin \theta = \frac{\pi}{2} \left[\frac{\pi^2}{4} + \frac{4}{\pi^2} + 1 \right]^{-1/2} \Rightarrow \theta = \sin^{-1}(0.80) = 52.96^\circ$$

Space for rough work — will NOT be submitted for evaluation

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